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DETERMINANTAL IDEALS FROM SYMMETRIZED SKEW TABLEAUX

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DETERMINANTAL IDEALS FROM SYMMETRIZED SKEW TABLEAUX

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
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By
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Lexington, Kentucky

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ABSTRACT OF DISSERTATION

DETERMINANTAL IDEALS FROM SYMMETRIZED SKEW TABLEAUX

We study a class of determinantal ideals called skew tableau ideals, which are generated by $t \times t$ minors in a subset of a symmetric matrix of indeterminates. The initial ideals have been studied in the 2×2 case by Corso, Nagel, Petrović and Yuen. Using liaison techniques, we have extended their results to include the original determinantal ideals in the 2×2 case, as well as special cases of the ideals in the $t \times t$ case. In particular, for any skew tableau ideal of this form, we have defined an elementary biliaison between it and one with a simpler structure. Repeated applications of this result show that these skew tableau ideals are glicci, and thus Cohen-Macaulay.

A number of other classes of ideals have been studied using similar techniques, and these depend on a technical lemma involving determinantal calculations. We have uncovered an error in this result, and have used the straightening law for minors of a matrix to establish a new determinantal relation. This new tool fixes the gaps in the previous papers and is a critical step in our own analysis of skew tableau ideals.

KEYWORDS: commutative algebra, determinantal ideal, liaison, skew tableau

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S.D.G.

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Chapter 1 Introduction

The powerful interactions between algebra and geometry which Descartes and Fermat began exploring almost 400 years ago have matured beautifully into modern algebraic geometry. At the end of the 19th century, the heavy dependence on intuition was increasingly problematic. David Hilbert's program of formalization pointed the way to greater rigor, and the tools of commutative algebra developed by Emmy Noether were eventually established as the foundation of algebraic geometry by mathematicians such as Oscar Zariski and André Weil. In the second half of the last century, Serre and Grothendieck propelled the subject to further heights of sophistication with the concepts of sheaves and schemes.

The history of Liaison Theory (or Linkage Theory) follows similar contours, with roots in the work of Francesco Severi nearly a hundred years ago and of his student Federico Gaeta in the 1950s. Gaeta determined that space curves with finite residual are exactly the arithmetically normal curves; these notions were studied in [19] by Peskine and Szpiro in 1974, when the modern treatment of Liaison Theory may be properly said to have begun. Their formulation of Gaeta's Theorem is that the codimension 2 subschemes that can be linked to a complete intersection are exactly the arithmetically Cohen-Macaulay ones. This is actually a fact about determinantal schemes, since every standard determinantal scheme is arithmetically Cohen-Macaulay, and in codimension 2 the notions are equivalent.

Much work has been done to extend and generalize Gaeta's Theorem (e.g. in [15, 11, 19]), and liaison has been used as a tool in a wide variety of contexts (e.g. in [12, 4]). Recently, Gorla and others have studied the G-liaison classification of different types of determinantal schemes, defined by ideals associated to minors of mixed size in various subsets of a matrix of indeterminates. Our main results extend this line of study by examining the liaison classification of skew-tableau ideals, which were introduced in [4]. We have also proven a determinantal identity that is useful in producing elementary biliaisons. This fills a gap in the proof of a lemma that is used in several papers on the linkage of determinantal ideals.

Linkage of Determinantal Ideals

The main object of study in Liaison Theory is links between schemes and between the ideals that define them. Roughly speaking, two varieties (e.g. algebraic curves, surfaces, etc.) are linked if their union is nice enough; when this happens, properties of one variety carry over to the other. For us, "nice enough" means Gorenstein, and the notions are *G-links* and *G-liaison*. When an ideal is in the G-liaison class of a complete intersection, we say that it is *glicci*. A major result in Liaison Theory [19] is that every glicci ideal is Cohen-Macaulay; it is still an open question whether the converse is true.

To determine whether a given ideal is glicci, one must produce a sequence of links that eventually lead to a complete intersection. Our main method of producing links is an *elementary biliaison*. In [15] it was shown that if there is an elementary biliaison between ideals I and J, then they can be G-linked in two steps.

There are several classes of determinantal ideals that have been studied using liaison techniques (such as in [3, 5, 8, 12, 15]). These are generated by minors of a designated size in the given shape:

- matrix determinantal ideals
- symmetric matrix determinantal ideals
- ladder determinantal ideals
- two-sided mixed ladder determinantal ideals
- symmetric ladder determinantal ideals

In every one of these classes, the ideal is glicci, its generating set of minors forms a Gröbner basis, and its initial ideal is squarefree monomial and glicci.

In the same vein, we study the liaison classification of a class of ideals called *skew tableau ideals*. If $A = (x_{i,j})$ is a matrix of indeterminates, then we define a *skew tableau* F to be a subset of A above the main diagonal such that if $x_{i,j}$ and $x_{i,k}$ are in F, then so is the rectangular submatrix with $x_{i,j}$ and $x_{i,k}$ as lower corners and $x_{1,j}$ and $x_{1,k}$ as upper corners. A *symmetric skew tableau* $\tilde{F}$ is a skew tableau reflected across the diagonal in a symmetric matrix of indeterminates. A *skew tableau ideal* is generated by minors of specified sizes in a skew tableau.

The following are examples of a skew tableau and a reflected skew tableau:

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$
		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
			$x_{3,4}$	$x_{3,5}$
			$x_{4,4}$	$x_{4,5}$

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$
$x_{1,2}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$	$x_{3,6}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$	$x_{4,5}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$	$x_{4,5}$		
$x_{1,6}$	$X_{2,6}$	$X_{3,6}$			

Ideals generated by 2×2 minors in skew tableaux were introduced in [4], where they arose in connection with edge ideals of Ferrers graphs. It was shown in that paper that the generating set forms a Gröbner basis, and that their initial ideals are glicci. Our work uses this result as a starting point.

In Section 2.3, we discuss a gap in the proof of a lemma from [10]. The lemma involves determinantal calculations that are used in showing elementary biliaisons in several of the linkage results for determinantal ideals mentioned above. We explain the error in the proof, and then prove a different determinantal identity that implies a strengthened version of the lemma. We apply this new version of the lemma in Chapter 4.

In Chapter 3, we show that skew tableau ideals generated by 2×2 minors are glicci, and thus Cohen-Macaulay. We do this by producing an elementary biliaison between an arbitrary skew tableau ideal and one generated in a strictly smaller skew tableau. Repeating this process a finite number of times eventually links to an ideal generated by indeterminates, which is a complete intersection.

In Chapter 4, we study skew tableau ideals generated by higher order minors. If there are no "holes," these ideals are a special case of symmetric ladder ideals. We recall some properties of these ideals from other work, reprove a Gröbner basis calculation and codimension formula, and give a new proof that these ideals are glicci (which does not depend on the faulty lemma mentioned above). For unsymmetrized skew tableau ideals, we use some of our determinantal calculations from Section section 2.3 to show that the minors form a Gröbner basis. We also offer some examples to show that for arbitrary skew tableau ideals, the Gröbner basis property and primality may not longer hold. We conclude with some plans for further study.

Chapter 2 Preliminaries

2.1 Standard Results in Liaison Theory

Liaison theory, or linkage, is typically regarded as a classification tool for ideals. Among other nice properties, linkage preserves Cohen-Macaulayness. Though there are many structural results established for complete intersection-linkage (i.e. CI-linkage), less is known about Gorenstein linkage (i.e. G-linkage). Of particular interest is showing that a various classes of ideals are *glicci*, i.e. in the Gorenstein liaison class of a complete intersection. Every glicci ideal is Cohen-Macaulay; a significant open question is whether the converse is true.

We present here an overview of some of the basic definitions and constructions in liaison theory, along with some classical results which we use in our own results. We assume throughout that K is an algebraically closed field, and that $R = K[x_0, \ldots, x_n]$ is a polynomial ring over K.

Definition 2.1.1. Let $J \subset R$ be a homogeneous, saturated ideal. We say that J is *Gorenstein in dimension* $\leq c$ if the localization $(R/J)_p$ is a Gorenstein ring for any prime ideal p of R/J of height smaller than or equal to c. We often say that J is G_c. We call *generically Gorenstein*, or G_0, an ideal J which is Gorenstein in codimension 0.

Definition 2.1.2. An ideal $I \subset R$ is said to be (directly) *linked* to an ideal $J \subset R$ by a Gorenstein ideal $\mathfrak{c} \subset R$ if $\mathfrak{c} \subset I \cap J$, and $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$.

Recall that the ideal quotient corresponds to set difference for algebraic varieties. If V and W are affine varieties with ideals $I(V)$ and $I(W)$, then $I(V) : I(W) = I(V \backslash W)$. Thus we may say heuristically that two ideals are linked when the geometric objects they describe have a nice union.

To show that a given ideal I is glicci, we must produce a sequence of new ideals that eventually link to a complete intersection. There are two classical constructions that are useful for producing these links.

Definition 2.1.3. Let $A \subset B \subset R$ be homogeneous ideals such that $ht(A) = ht(B) - 1$ and R/A is Cohen-Macaulay. Let $f \in R_d$ be a homogeneous element of degree d such that $A : f = A$. The ideal $C := A + fB$ is called a *Basic Double Link* of degree d of B on A. If moreover A is G_0 and B is unmixed, then C is a *Basic Double G-Link* of B on A.

Definition 2.1.4. Let $I, J \subset R$ be homogeneous, saturated, unmixed ideals, such that $ht(I) = ht(J) = c$. We say that J is obtained by an *elementary biliaison* of height l from I if there exists a Cohen-Macaulay ideal N in R of height $c-1$ such that $N \subset I \cap J$ and $J/N \cong I/N\,(-l)$ as R/N modules. If in addition the ideal N

is G_0, then J is obtained from I via an *elementary G-biliaison.* If $l > 0$ we have an *ascending* elementary G-biliaison.

Basic Double Linkage is a special case of biliaison. Moreover, it can be shown that if J is obtained from I via an elementary biliaison of height l on N, then there exists an ideal H and a $d \in \mathbb{Z}$ s.t. H is a Basic Double Link of degree $d + l$ of I on N and also a Basic Double Link of degree d of J on N.

Theorem 2.1.5 ([15])**.** *Suppose J is obtained from I via an elementary G-biliaison. Then I is linked to J in two steps.*

We end this section with a classical result of Peskine and Szpiro that reveals some of the power of linkage.

Theorem 2.1.6 ([19])**.** *Suppose I and J are linked by a Gorenstein ideal N. Then R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay.*

In particular, it follows from this result that glicci ideals are Cohen-Macaulay. Although an affirmative answer has been shown in special cases (e.g. Gaeta's Theorem for codimension 2), it is unknown in general whether every Cohen-Macaulay ideal is glicci.

2.2 Initial Ideals, Gröbner Bases, and Vertex Decomposibility

Again we let R be a standard graded polynomial ring in finitely many indeterminates over an algebraically closed field K. Gröbner bases are a powerful computation tool for studying ideals in commutative algebra and algebraic geometry, and we will use them several times throughout this paper.

For a fixed term-order $<$ on R, recall that the initial term (or leading term) $in_<(r)$ of an element $r \in R$ is the largest monomial of r with respect to $<$. If $I \subset R$ is an ideal, then the *initial ideal* of I, denoted $in_<(I)$, is the ideal of R generated by the initial terms of the elements of I,

$$in_<(i) = \langle in(r) \in R \,|\, in(r) \in I \rangle.$$

When the term-order is clear, we will just write $in(r)$ and $in(I)$.

Definition 2.2.1. Let X be a matrix of indeterminates. Let $<$ be a term-order on the set of terms of $K[X]$. The term-order $<$ is *diagonal* if the leading term with respect to $<$ of the determinant of any submatrix of X is the product of the indeterminates on its diagonal.

If $X = (x_{i,j})$, then the lexicographic term-order on monomials is a well-known example of a diagonal term-order.

Definition 2.2.2. Let $\mathfrak{G}$ be a set of polynomials in R that generate an ideal I. Then $\mathfrak{G}$ is a *Gröbner basis* of I if the initial terms of the elements of $\mathfrak{G}$ generate $in(I)$.

Every ideal of R has a Gröbner basis by Hilbert's basis theorem, and Buchberger's algorithm is a standard method for transforming a generating set of an ideal into a Gröbner basis. We will use the well-known Buchberger's Criterion and two lemmas from [2] for our Gröbner basis calculations.

Definition 2.2.3. Let $f, g \in R$ be nonzero polynomials. The *S-polynomial* of f and g is the polynomial

$$S(f,g) = \frac{\operatorname{lcm}(in(f), in(g))}{in(f)} \cdot f - \frac{\operatorname{lcm}(in(f), in(g))}{in(g)} \cdot g$$

where lcm refers to the least common multiple.

Theorem 2.2.4 (Buchberger's Criterion)**.** *Let I be a polynomial ideal. Then a basis $\mathfrak{G} = \{g_1, \ldots, g_t\}$ for I is a Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder of division of $S(g_i, g_j)$ by $\mathfrak{G}$ is zero.*

Lemma 2.2.5 ([2, Lemma 1.3.13])**.** *Let $I \subset K[X]$ be an ideal with Gröbner bases G. Let $F \subset X$ have the property that if $f \in G$ and $in(f) \in K[F]$, then $f \in K[F]$. Then $G \cap K[F]$ is a Gröbner basis of $I \cap K[F]$.*

Lemma 2.2.6 ([2, Lemma 1.3.14])**.** *Let $I, J \subset K[x_1, \ldots, x_r]$ be homogeneous ideals. Let G be a Gröbner basis of I and let H be a Gröbner basis of J. Then $G \bigcup H$ is a Gröbner bases of $I + J$ if and only if for all $f \in F$ and $g \in G$, there is an $h \in I \cap J$ such that $in(h) = \operatorname{lcm}(in(f), in(g))$.*

Although it is not our main object of interest in this paper, linkage results for the initial ideals we study also provide information about their associated simplicial complexes (see [12]). The rest of this section may be skipped by readers uninterested in these results.

Definition 2.2.7. A *simplicial complex* Δ on $n+1$ vertices is a collection of subsets of $[n]$ such that for any $F \in \Delta$, if $G \subset F$, then $G \in \Delta$. An $F \in \Delta$ is called a *face* of Δ. The dimension of a face F is $\dim F = |F| - 1$, and the dimension of the complex is

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\}.$$

The complex $\Delta = 2^{\{0,\ldots,n\}}$ is called a *simplex*.

The *vertices* of Δ are the subsets of $[n]$ of cardinality one. The faces of Δ which are maximal with respect to inclusion are called *facets*. A complex is *pure* if all its facetrs have dimension equal to the dimension of the complex.

Notation 2.2.8. To each face $F \in \Delta$ we associate the following two simplicial subcomplexes of Δ: the *link* of F:

$$\operatorname{lk}_\Delta = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\}$$

and the *deletion*

$$\Delta - F = \{G \in \Delta \mid F \cap G = \emptyset\}.$$

If $F = \{k\}$ is a vertex, we denote the link of F and the deletion by $\mathrm{lk}_k(\Delta)$ and $\Delta - k$, respectively.

Definition 2.2.9. A simplicial complex Δ is *vertex decomposable* if it is a simplex, or it is the empty set, or there exists a vertex k such that $\mathrm{lk}_k(\Delta)$ and $\Delta - k$ are both pure and vertex decomposable, and

$$\dim \Delta = \dim(\Delta - k) = \dim \mathrm{lk}_k(\Delta) + 1.$$

These notions for simplices are relevant to our study of determinantal ideal because of the following construction.

Definition 2.2.10. The *Stanley-Reisner ideal* associated to a complex Δ on $n+1$ vertices is the squarefree monomial ideal

$$I_\Delta = (x_{i_1}, \ldots, x_{i_s} \mid \{i_1, \ldots, i_s\} \notin \Delta) \subset K[x_0, \ldots, x_n].$$

Conversely, to every squarefree monomial ideal $I \subset K[x_0, \ldots, x_n]$ one can associate the unique simplicial complex $\Delta(I)$ on $n+1$ vertices such that $I_{\Delta(I)} = I$.

2.3 A New Determinantal Identity

We discuss now an identity that will be a key for establishing the existence of certain elementary biliaisons later on. In particular, the identity will allow us to give a corrected proof of [10, Lemma 2.6]. There are several papers, including [8], [9], and [11], which employ the latter result. We begin with some notation drawn from [20].

Notation 2.3.1. Let M be an $m \times n$ matrix with entries in a commutative ring with identity. For ordered lists $A = \{a_1 < \cdots < a_s\}$ with $a_i \in [m]$ and $B = \{b_1 < \cdots < b_t\}$ with $b_i \in [n]$, we define $M_{A|B}$ to be the submatrix of M whose entries are have row indices in A and column indices in B (in the given order). Set $M(A|B) = \det M_{A|B}$ if $|A| = |B|$, and put $M(A|B) = 0$ otherwise.

We adopt the convention that the determinant of a 0×0 matrix is 1.

Definition 2.3.2. Let M be a square $n \times n$ matrix with A and B as above. Let $\tilde{A}$ be the complement of A in $[m]$ and let $\tilde{B}$ be the complement of B in $[n]$, and denote by $\sum A$ and $\sum B$ the sum of the elements of A and B respectively. We define the *Laplace product* to be $M\{A|B\} = (-1)^{\sum A + \sum B} M(A|B) M(\tilde{A}|\tilde{B})$.

We want to manipulate ordered lists.

Definition 2.3.3. For $\alpha \in A = \{a_1 < \cdots < a_s\}$, define

$$A - \alpha = \{a_1, \ldots, \widehat{\alpha}, \ldots, a_s\} = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s\}$$

if $\alpha = a_i$. For an integer $\beta \notin [A]$, we denote by $A + \beta$ the ordered list consisting of β and the entries of A. In particular, $|A+\beta| = |A|+1$. For example, $\{1, 5\}+4 = \{1, 4, 5\}$.

The goal of this section is to establish:

Lemma 2.3.4 ([10, Lemma 2.6]). *Let M be a matrix of size $m \times n$. Let I and K be ordered lists in $[m-1]$, and let J and L be ordered lists in $[n-1]$ with $|I| = |J| = |K| = |L| = t-1$. Let I_Y be the ideal generated by the t-minors of M of size $t \times t$ that do not involve both row m and column n. Then*

$$M(I+m|J+n) \cdot M(K|L) - M(I|J) \cdot M(K+m|L+n) \in I_Y.$$

Remark 2.3.5. In [10], the previous lemma is stated for symmetric $m \times m$ matrices, and the ideal I_Y is taken to be generated by all minors which do not involve the last row of M. If M is symmetric then this definition of I_Y is equivalent to the one we have given; however, that statement does not generalize in the case of a generic matrix. For its application here (and in other papers), we need our more general statement.

The proof of Lemma 2.3.4 depends on the following lemma of the same paper:

Lemma 2.3.6 ([10, Lemma 2.4]). *Let M be a matrix of size $m \times m$. Let I and K be lists of row indices and let J and L be lists of column indices with $|I| = |J| = |K| = |L| = t$. Let $I_{t+1}(M)$ be the ideal generated by the minors of M of size $(t+1) \times (t+1)$. Then*

$$M(I|J) \cdot M(K|L) - M(K|J) \cdot M(I|L) \in I_{t+1}(M).$$

We have uncovered a gap in the proof of [10, Lemma 2.4], and though its claim may still be true, we have not searched for another proof. Instead, we will establish a new determinantal identity using the straightening law for minors of a matrix that implies [10, Lemma 2.6] directly. Before we prove this new identity, we will explain the error in the proof of [10, Lemma 2.4].

Gap in Proof of [10, Lemma 2.4]

In the first half of the proof of this lemma, there is a chain of equalities that gives

$$M(I|J) \cdot M(K|L) - M(K|J) \cdot M(I|L)$$

as a summation of the terms

$$\begin{aligned} &M(i_1, \ldots, i_a | j_1, \ldots, j_b, l_{b+1}, \ldots, l_a) \cdot M(i_1, \ldots, i_{a-1}, k_a | l_1, \ldots, l_b, j_{b+1}, \ldots, j_a) \\ &\quad - M(i_1, \ldots, i_a | j_1, \ldots, j_{b-1}, l_b, \ldots, l_a) \cdot M(i_1, \ldots, i_{a-1}, k_a; l_1, \ldots, l_{b-1}, j_b, \ldots, j_a), \end{aligned}$$

where b ranges from 1 to a. After this, it is stated that "all the minors in the expression have all the rows and columns in common, except for possibly one. So Sylvester's Theorem applies, and the thesis follows." However, we have included below the only formulation of Sylvesters Identity known to us, and this does not apply to the expression in question. A more detailed discussion of Sylvester's Identity may be found in [1].

Proposition 2.3.7 (Sylvester's Identity)**.** *Let M be an $m \times n$ matrix. Let I be an ordered list in $[m-1]$, and let J an ordered list in $[n-1]$, with $|I| = |J| = t-1$. Then, for every $i \in I$ and $j \in J$,*

$$\begin{aligned} &M(I|J) \cdot M(I-i+m|J-j+m) - M(I-i+m|J) \cdot M(I|J-j+m) \\ &\qquad\qquad = M(I-i|J-j) \cdot M(I+m|J+n) \end{aligned}$$

Note that, for $t = 2$, this is the formula for computing the determinant of the 2×2 matrix $M_{I+m|J+n}$.

In general, Sylvester's identity relates an expression of four matrix determinants of size $t \times t$ to a product of an $(t-1) \times (t-1)$ determinant and a $(t+1) \times (t+1)$ determinant. All six of these minors have $(t-1)$ rows and columns in common. However, in Gorla's Lemma 2.4, the expression involves minors with $(t-1)$ rows in common but completely disjoint columns, and so Sylvester's Identity does not apply. A similar problem occurs in the second half of the proof, where the expressions involve minors with $(t-1)$ columns in common but completely disjoint rows.

In fact, not only does Sylvester's Identity not apply in this case, but the expressions

$$M(I|J) \cdot M(I-i+m|J-j+m) - M(I-i+m|J) \cdot M(I|J-j+m)$$

are not in general in the ideal $I_{t+1}(M)$ from Lemma 2.3.6.

Example 2.3.8. Let M be the 6×6 matrix:

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} & x_{3,6} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} & x_{4,6} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & x_{5,6} \\ x_{6,1} & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} \end{pmatrix}.$$

Let $(i_1, i_2, i_3) = (1, 2, 3)$, $(j_1, j_2, j_3) = (1, 2, 3)$, $(k_1, k_2, k_3) = (4, 5, 6)$, and $(l_1, l_2, l_3) = (4, 5, 6)$. Then an example of one of the summands in the proof of Lemma 2.4 is

$$\begin{vmatrix} x_{1,1} & x_{1,5} & x_{1,6} \\ x_{2,1} & x_{2,5} & x_{2,6} \\ x_{3,1} & x_{3,5} & x_{3,6} \end{vmatrix} \cdot \begin{vmatrix} x_{1,4} & x_{1,2} & x_{1,3} \\ x_{2,4} & x_{2,2} & x_{2,3} \\ x_{6,4} & x_{6,2} & x_{6,3} \end{vmatrix} - \begin{vmatrix} x_{1,4} & x_{1,5} & x_{1,6} \\ x_{2,4} & x_{2,5} & x_{2,6} \\ x_{3,4} & x_{3,5} & x_{3,6} \end{vmatrix} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{6,1} & x_{6,2} & x_{6,3} \end{vmatrix}.$$

However, according to Macaulay2, the expression is not in $I_4(M)$.

Lemma 2.4 claims that the expression

$$\begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix} \cdot \begin{vmatrix} x_{4,4} & x_{4,5} & x_{4,6} \\ x_{5,4} & x_{5,5} & x_{5,6} \\ x_{6,4} & x_{6,5} & x_{6,6} \end{vmatrix} - \begin{vmatrix} x_{4,1} & x_{4,2} & x_{4,3} \\ x_{5,1} & x_{5,2} & x_{5,3} \\ x_{6,1} & x_{6,2} & x_{6,3} \end{vmatrix} \cdot \begin{vmatrix} x_{1,4} & x_{1,4} & x_{1,4} \\ x_{2,5} & x_{2,5} & x_{2,5} \\ x_{3,6} & x_{3,6} & x_{3,6} \end{vmatrix} \in I_4(M).$$

Macaulay2 verifies this claim; however, the presentation of the expression that is given in terms of generators of $I_4(M)$ requires 15 generators, and not all of the coefficients given are minors of M. For this reason, we have not searched for a different lemma comparable to Lemma 2.4.

Determinantal Lemma

Our proof of Proposition 2.3.10 uses the following result by Swan.

Theorem 2.3.9 ([20, Theorem 2.6])**.** *Let M be an $m \times m$ matrix. Then, for every choice of ordered lists A and B in $[m]$,*

$$\sum_{V \subseteq B} M\{A|V\} = \sum_{U \supseteq A} M\{U|B\}.$$

We use this result to show:

Proposition 2.3.10. *Let M be a matrix of size $(2t-1) \times (2t-1)$. Let I and K as well as J and L disjoint ordered lists of $t-1 \geq 1$ numbers such that*

$$I \cup K = J \cup L = [2t-2].$$

Then we have

$$\begin{aligned} M(I+m|J+n) \cdot M(K|L) - M(I|J) \cdot M(K+m|L+n) \\ = \sum_{i \in I} (-1)^{n+i} M(I-i+m|J) \cdot M(K+i|L+n) \\ - \sum_{l \in L} (-1)^{n-l} M(I+m|J+l) \cdot M(K|(L-l+n), \end{aligned}$$

where $m = n = 2t-1$.

Proof. We apply Theorem 2.3.9 to M with $A = K$ and $B = L+n$ as the fixed subsets:

$$\sum_{V \subseteq L+n} M\{K|V\} = \sum_{U \supseteq K} M\{U|L+n\}.$$

On the left hand side, the only subsets $V \subseteq L+n$ for which the Laplace products $M\{A|V\}$ are nonzero are those with size $|V| = |A| = t-1$. Since M is a $(2t-1) \times (2t-1)$ matrix, these $M\{K|V\}$ are the signed products of a $t \times t$ minor and a $(t-1) \times (t-1)$ minor. Similarly, on the right hand side, the only subsets $U \supseteq K$ for which the Laplace products $M\{U|L+n\}$ are nonzero are those with size $|U| = |B| = t$. In this case, these $M\{U|L+n\}$ are signed products of a $t \times t$ minor and a $(t-1) \times (t-1)$ minor. Thus, we get the equality

$$\sum_{l \in L} M\{K|L-l+n\} + M\{K|L\} = \sum_{i \in I} M\{K+i|L+n\} + M\{K+m|L+n\},$$

which can be expanded as

$$\sum_{l\in L}(-1)^{n-l}M(K|L-l+n)\cdot M(I+m|J+l)+M(K|L)\cdot M(I+m|J+n)$$
$$=\sum_{i\in I}(-1)^{n+i}M(K+i|L+n)\cdot M(I-i+m|J)$$
$$+(-1)^{m+n}M(K+m|L+n)\cdot M(I|J).$$

The signs of these minors come directly from Definition 2.3.2 after we multiply both sides by $(-1)^{\sum K+\sum L}$. Since $m=n$, the claim follows. □

As a consequence, we get a slightly strengthened version of [10, Lemma 2.6], i.e. Lemma 2.3.4.

Corollary 2.3.11. *Let M be a matrix of size $m\times n$. Let I and K be ordered lists in $[m-1]$, and let J and L be ordered lists in $[n-1]$, where $|I|=|J|=|K|=|L|=t-1$. Let I_Y be the ideal that is generated by the t-minors of the submatrix of M with column indices in $(I\cup K)+m$ and row indices in $(J\cup L)+n$ that do not involve both row m and column n. Then*

$$M(I+m|J+n)\cdot M(K|L)-M(I|J)\cdot M(K+m|L+n)\in I_Y.$$

Proof. Consider first the special case where I and K as well as J and L are disjoint and

$$I\cup K=J\cup L=[2t-2].$$

Then the claim follows by Proposition 2.3.10 because each of the t-minors $M(K+i|L+n)$ and $M(I+m|J+l)$ is in I_Y.

Now we reduce the general case to the above special case. Consider any lists I,J,K, and L as in the statement. Write $I=\{i_1<\cdots<i_{t-1}\}$, $J=\{j_1<\cdots<j_{t-1}\}$, $L=\{k_1<\cdots<k_{t-1}\}$, and $L=\{l_1<\cdots<l_{t-1}\}$. Form a $(2t-1)\times(2t-1)$ submatrix N of M whose entries are taken from the indicated rows and columns of M:

$$N=M_{(i_1,\ldots,i_{t-1},k_1,\ldots,k_{t-1},m)|(j_1,\ldots,j_{t-1},l_1,\ldots,l_{t-1},n)}.$$

Applying the above special case to the submatrix N, our claim follows. □

Example 2.3.12. Let $M=(x_{i,j})$ be a 5×5 matrix, and let S_Y be the ideal generated by the 3×3 minors of M that do not involve the entry $x_{5,5}$. Let $I=(3,4)$, $J=(1,2)$, $K=(1,2)$, and $L=(3,4)$. Then according to Corollary 2.3.11 ,

$$M(1,2|3,4)\cdot M(3,4,5|1,2,5)-M(1,2,5|3,4,5)\cdot M(3,4|1,2)\in S_Y.$$

This follows from Proposition 2.3.10, which gives

$$\{3,4,5|1,2,5\}-\{1,2,5|3,4,5\}$$
$$=\{1,2,3|3,4,5\}+\{1,2,4|3,4,5\}-\{3,4,5|1,2,3\}-\{3,4,5|1,2,4\}.$$

More explicitly, this is the identity

$$\begin{aligned}
&\begin{vmatrix} x_{3,1} & x_{3,2} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,5} \end{vmatrix} \cdot \begin{vmatrix} x_{1,3} & x_{1,4} \\ x_{2,3} & x_{2,4} \end{vmatrix} - \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ x_{5,3} & x_{5,4} & x_{5,5} \end{vmatrix} \cdot \begin{vmatrix} x_{3,1} & x_{3,2} \\ x_{4,1} & x_{4,2} \end{vmatrix} \\
&\qquad = \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,3} & x_{3,4} & x_{3,5} \end{vmatrix} \cdot \begin{vmatrix} x_{4,1} & x_{4,2} \\ x_{5,1} & x_{5,2} \end{vmatrix} - \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ x_{4,3} & x_{4,4} & x_{4,5} \end{vmatrix} \cdot \begin{vmatrix} x_{3,1} & x_{3,2} \\ x_{5,1} & x_{5,2} \end{vmatrix} \\
&\qquad\quad - \begin{vmatrix} x_{3,1} & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & x_{4,3} \\ x_{5,1} & x_{5,2} & x_{5,3} \end{vmatrix} \cdot \begin{vmatrix} x_{1,4} & x_{1,5} \\ x_{2,4} & x_{2,5} \end{vmatrix} + \begin{vmatrix} x_{3,1} & x_{3,2} & x_{3,4} \\ x_{4,1} & x_{4,2} & x_{4,4} \\ x_{5,1} & x_{5,2} & x_{5,4} \end{vmatrix} \cdot \begin{vmatrix} x_{1,3} & x_{1,5} \\ x_{2,3} & x_{2,5} \end{vmatrix}.
\end{aligned}$$

We illustrate this relation by shading the subregion of M corresponding to each minor. The equality above comes from the determinantal identity in Proposition 2.3.10, which gives:

$$
\begin{aligned}
&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \\
-&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \\
+&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \\
=&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \\
-&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \\
+&\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{pmatrix}
\end{aligned}
$$

The reader should be reminded of calculating a determinant using expansion by cofactors. Indeed, this is the same process, but using a more general notion of cofactor.

2.4 Skew Tableau Ideals

Our main object of study is ideals generated by minors in a subregion of a matrix of indeterminates called a skew tableau. In this section, we will introduce the definitions and notation for describing skew tableau ideals. In Section 3.1 we recall some known results about skew tableau ideals generated by 2×2 minors, and in Section 3.2 we show that these ideals are glicci. In Chapter 4 we discuss skew tableau ideals generated by higher order minors.

Definition 2.4.1. Let $X \subset \mathbb{N}^2$ represent a rectangular array of boxes, with the box in row i and column j indicated by its location $(i,j) \in X$. A *skew tableau* F is a subregion of X such that if F contains two boxes on the same row, say the boxes at (i,p) and (i,q) with $p \leq q$, then it also contains all the boxes in the largest rectangular subregion of X with lower corners at (i,p) and (i,q), i.e.

$$\begin{matrix} (1,p) & \dots & (1,q) \\ \vdots & \ddots & \vdots \\ (i,p) & \dots & (i,q) \end{matrix}$$

We will always label tableaux so that, in the notation above, $p \geq i$, which places F above the main diagonal of X. The reasons for this will be made clear in Definition 2.4.4. Eventually, we will take X to be a symmetric matrix of indeterminates.

Notation 2.4.2. According to the previous definition, a finite list of these pairs of locations (i,p) and (i,q) determines a skew tableau F. However, sometimes it will be convenient to define the left and right borders of the tableau separately. We do this by a list of left corners $(b_1, c_1), \dots, ((b_\ell, c_\ell)$ and a list of right corners $(a_1, d_1), \dots, (a_k, d_k)$ that satisfy the following inequalities:

$$\begin{aligned} &1 \leq c_\ell \leq \cdots \leq c_1 \leq d_1 \leq \cdots \leq d_k \\ &1 \leq a_k \leq \cdots \leq a_1 \\ &1 \leq b_\ell \leq \cdots \leq b_1 \\ &b_i \leq c_i \\ &a_i \leq d_i \end{aligned}$$

These conditions ensure that left corners are indeed farther left than right corners, that both lists of corners are ordered from lowest to highest (i.e. from greatest to smallest row index), and that all corners are on or above the main diagonal of X. The skew tableau $F(a,b,c,d)$ defined by these corners is

$$F = \{(i,j) \in \mathbb{N}^2 \,|\, 1 \leq i \leq a_r, j \leq d_r \text{ for some } r \text{ and } 1 \leq i \leq b_s, j \geq c_s \text{ for some } s\}$$

A more concise way to express this information is to define vectors

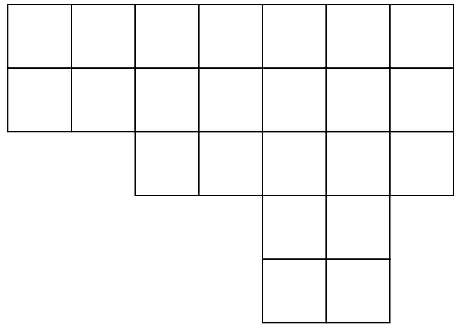
$$\begin{aligned} a &= (a_1, \dots, a_k) \\ b &= (b_1, \dots, b_\ell) \\ c &= (c_1, \dots, c_\ell) \\ d &= (d_1, \dots, d_k) \end{aligned}$$

and denote F as $T(a,b,c,d)$.

Example 2.4.3. The skew tableau with vectors

$$\begin{aligned} a &= (5,3) \qquad & b &= (5,3,2) \\ d &= (7,8) \qquad & c &= (6,4,2) \end{aligned}$$

would look like this:



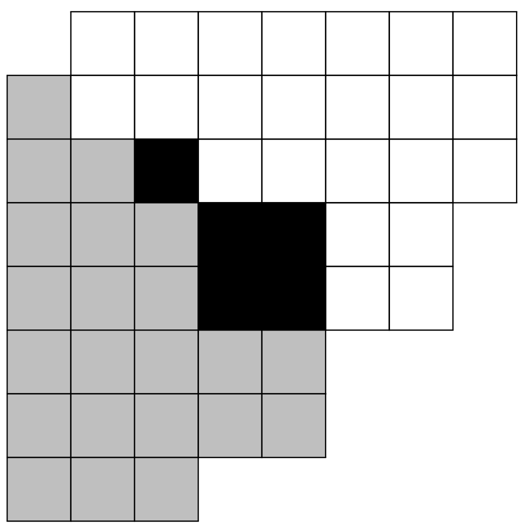
where the top left box is at location $(1, 2)$.

Definition 2.4.4. Let $F \subset X$ be a skew tableau, and recall that $a_i \leq c_i$, so that F includes no locations below the main diagonal of X. We define a *symmetrized skew tableau* $\tilde{F}$ to include every location (i, j) in F along with its transposed location (j, i), i.e.

$$\tilde{F} = \{(i, j) \in X \mid (i, j) \in F \text{ or } (j, i) \in F\}.$$

Graphically, this is the shape F along with its reflection across the main diagonal of X. We always assume that X is large enough to include all locations in $\tilde{F}$. At times we will colloquially refer to these as reflected skew tableaux.

Example 2.4.5. If F is the skew tableau in Example 2.4.3, then $\tilde{F}$ would look like this:



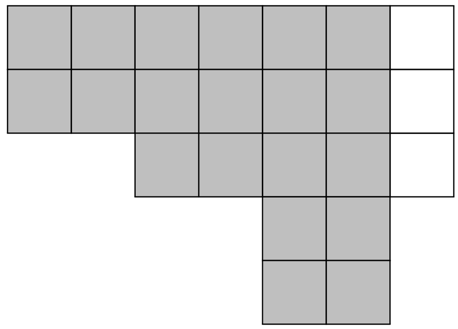
The white boxes form the original skew tableau F, the gray boxes are those added to form $\tilde{F}$, and the black boxes are "holes" in the skew tableau.

Definition 2.4.6. Suppose F has right corners $\{(a_i, d_i)\}$, with $1 \leq i \leq k$. Define F_i to be the intersection of F with the following subset of X:

$$X_i = \begin{pmatrix} (1,1) & \dots & (1, d_i) \\ \vdots & \ddots & \vdots \\ (a_i, 1) & \dots & (a_i, d_i) \end{pmatrix}$$

Each F_i contains exactly those locations of F which are above and to the left of the i^{th} right corner, inclusive. Note that $F = \bigcup_{i=1}^{k} F_i$ and $\tilde{F} = \bigcup_{i=0}^{k} \tilde{F}_i$.

Example 2.4.7. Using the same skew tableau F as in the previous examples, F_1 is the region shaded in gray:



In this paper, we are interested in ideals generated by minors in a skew tableau-shaped subset of a matrix of indeterminates. We will slightly abuse notation by now taking $X = (x_{i,j})$ to be a symmetric matrix of indeterminates in R with $x_{i,j}$ at location (i,j) when $i \leq j$ and $x_{j,i}$ at location (i,j) when $i > j$. From this point forward, all skew tableaux F and $\tilde{F}$ will be considered as subsets of such a matrix.

Definition 2.4.8. Let $F = \bigcup_{i=1}^{k} F_i$ be a skew tableau of indeterminates and let $t = (t_1, \ldots, t_k)$ a vector with positive integer entries. Define $I_{t_i}(F_i)$ to be the ideal generated by the $t_i \times t_i$ minors in F_i, and let $I_t(F) = I_{t_1}(F_1) + \ldots I_{t_k}(F_k)$. We call $I_t(F)$ a *skew tableau ideal.* A projective scheme associated with such an ideal is a *skew tableau scheme.* Similarly, we define a *symmetrized skew tableau ideal* to be $I_t(\tilde{F}) = I_{t_1}(\tilde{F}_1) + \ldots I_{t_k}(\tilde{F}_k)$ and call the corresponding projective scheme a *symmetrized skew tableau scheme.* Note that $I_t(F) \subset I_t(\tilde{F})$.

Example 2.4.9. We again let F be defined by the vectors

$$\begin{aligned} a &= (5,3) & \qquad b &= (5,3,2) \\ d &= (7,8) & \qquad c &= (6,4,2) \end{aligned}$$

As a skew tableau of indeterminates, this is:

$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$
$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$
		$x_{3,4}$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$
				$x_{4,6}$	$x_{4,7}$	
				$x_{5,6}$	$x_{5,7}$	

The symmetrized skew tableau $\tilde{F}$ is:

	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$
$x_{1,2}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$			$x_{4,6}$	$x_{4,7}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$			$x_{5,6}$	$x_{5,7}$	
$x_{1,6}$	$x_{2,6}$	$x_{3,6}$	$x_{4,6}$	$x_{5,6}$			
$x_{1,7}$	$x_{2,7}$	$x_{3,7}$	$x_{4,7}$	$x_{5,7}$			
$x_{1,8}$	$x_{2,8}$	$x_{3,8}$					

Note that there are no new variables introduced by symmetrizing.

Example 2.4.10. Let $\tilde{F}$ be the tableau

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$
$x_{1,2}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$		

Then $I_3(\tilde{F})$ is the ideal generated by the minors

$$\begin{vmatrix} x_{1,1} & x_{1,4} & x_{1,5} \\ x_{1,2} & x_{2,4} & x_{2,5} \\ x_{1,3} & x_{3,4} & x_{3,5} \end{vmatrix} \text{ and } \begin{vmatrix} x_{1,1} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,3} & x_{2,4} \\ x_{1,4} & x_{3,4} & x_{4,4} \end{vmatrix}.$$

Remark 2.4.11. If we define a skew tableau ideal from a tableau with a hole at location (i, j), we do not obtain the same ideal by including the variable $x_{i,j}$ but setting it equal to 0. For example, if we set $x_{1,1} = 0$ in the matrix below, then the result is that

$$\begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix} = -x_{1,2}x_{2,1}x_{3,3} + x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,1}x_{3,2} - x_{1,3}x_{2,2}x_{3,1},$$

when in fact we want the whole minor to be 0.

Remark 2.4.12. Note that if A is any $t_i \times t_i$ submatrix of $\tilde{F}_i$ and X_i is the submatrix of X defined in Definition 2.4.6, then either $A \in \tilde{F}_i \cap X_i$ or $A^{\mathrm{tr}} \in \tilde{F}_i \cap X_i$. So $I_{t_i}(\tilde{F}_i) = I_{t_i}(\tilde{F}_i \cap X_i)$. For instance, the two shaded minors below are equal.

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$
$x_{1,2}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$		

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$
$x_{1,2}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$		

We may ignore the last row of the tableau, and the ideal defined by minors in the remaining region will be the same. In general, if F has n rows, then we may ignore all rows below the n^{th} row of $\tilde{F}$.

It may easily occur for a skew tableau ideal that one of the subregions is redundant or trivial. For $I_t(F) = \sum_{i=1}^{k} I_{t_i}(F_i)$, if both $F_a \subset F_b$ and $t_b \le t_a$, then clearly $I_{t_a}(F_a) \subset I_{t_b}(F_b)$ and $I_t(F) = \sum_{i \ne b}^{k} I_{t_i}(F_i)$. In this case we will drop the i^{th} corner and subregion from the description of F and $I_t(F)$. We adopt the same convention for symmetrized skew tableau ideals. The following lemma shows two more simplifications we may assume for skew tableau ideals.

Lemma 2.4.13. *Let $F = T(a, b, c, d)$ be a skew tableau. We may assume, without loss of generality, that in a symmetrized skew tableau ideal $I_t(\tilde{F})$,*

(1.) $t_i \le a_i$ for all i, and

(2.) $d_b - d_a > t_b - t_a$ for all $a > b$. In particular, $d_{i-1} - d_i > t_{i-1} - t_i$ for all i.

Proof. (1.) If $t_i > a_i$, then $I_{t_i}(\tilde{F}_i)$ is generated by $t_i \times t_i$ minors in a subregion of $\tilde{F}$ with less than t_i rows, and so $I_{t_i}(\tilde{F}_i) = (0)$. Suppose $a = (a_1, \ldots, a_k)$ and $a = (d_1, \ldots, d_k)$. Then we may replace F by $F' = T(a', b, c, d')$, where $a' = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$ and $d' = (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k)$. We let $t' = (t_1, \ldots, t_{i-1}, \ldots, t_k)$. Then

$$I_t(\tilde{F}) = \sum I_{t_r}(\tilde{F}_r) = \sum_{r \ne i} I_{r_i}(\tilde{F}_r) = I_{t'}(\tilde{F}').$$

(2.) Suppose $d_b - d_a \le t_b - t_a$ for some $a > b$. Recall from Remark 2.4.12 that $I_{t_1}(\tilde{F}_i \cap X_i) = I_{t_i}(\tilde{F}_i)$, and note that $\tilde{F}_b \cap X_b$ involves exactly $d_b - d_a$ columns that are not in $\tilde{F}_a \cap X_a$. Suppose A is a $t_b \times t_b$ submatrix of $\tilde{F}_b \cap X_b$. Then A must have at least $t_b - (d_b - d_a)$ columns in $\tilde{F}_a \cap X_a$, and since $t_b - (d_b - d_a) \ge t_a$, it follows that $\det(A) \in I_{t_a}(\tilde{F}_a \cap X_a)$. Thus $I_{t_b}(\tilde{F}_b) \subset I_{t_b}(\tilde{F}_b)$, and we may drop the b^{th} pair of right corners to form a new skew tableau F', and replace $I_t(\tilde{F})$ by an ideal $I_{t'}(\tilde{F}')$ with the property that $d_b - d_a > t_b - t_a$ for all $a > b$. $\square$

Unsymmetrized skew tableau ideals are a special case of symmetrized skew tableau ideals. In this paper we will prove results in the symmetrized case, and the corresponding results will follow for unsymmetrized tableaux.

Remark 2.4.14. Any skew tableau ideal $I_t(F)$ may be regarded as a symmetrized skew tableau ideal. Suppose $F = F(a, b, c, d)$ has n rows and k corners, and let $\delta = n - c_k + 1$. Then we may define a new tableau $F' = (a, b, c+\delta, d+\delta)$, where $c+\delta$ and $d+\delta$ represent the vectors obtained by adding δ to every component of c and d, respectively. After relabelling the variables of F' by $x_{i,j} \to x_{i,j-\delta}$, we have that $I_t(F) = I_t(F')$, and since symmetrizing F' does not add any new minors, it follows that $I_t(F) = I_t(F') = I_t(\tilde{F'})$. For instance, suppose F is the tableau

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$
	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$
			$x_{3,4}$	$x_{3,5}$	
			$x_{4,4}$	$x_{4,5}$	

Then $\tilde{F'}$ would be the tableau

				$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$
					$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$
							$x_{3,4}$	$x_{3,5}$	
							$x_{4,4}$	$x_{4,5}$	
$x_{1,1}$									
$x_{1,2}$	$x_{2,2}$								
$x_{1,3}$	$x_{2,3}$								
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$						
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$	$x_{4,5}$						
$x_{1,6}$	$x_{2,6}$								

By Remark 2.4.12, we may ignore the part of the tableau $\tilde{F'}$ that is below row 4 when considering $I_t(\tilde{F'})$, which means that $I_t(F) = I_t(\tilde{F'})$, and thus that skew tableau ideals are a special case of symmetrized skew tableau ideals.

Chapter 3 Skew Tableau Ideals Generated by 2×2 Minors

3.1 Known Results

In [4] skew tableau ideals generated by 2×2 minors are introduced for the study of blowups of specialized Ferrers ideals. Among the other things, they show that the initial ideals are Cohen-Macaulay and glicci. This leads naturally to the question of whether these results generalize to the full ideals, and to the case of minors of higher order. Before we address these questions, we recall some important results proven in [4].

Theorem 3.1.1 ([4, Theorem 2.4]). *The 2-minors of symmetrized skew tableau ideals form a Gröbner basis with respect to the lexicographic term-order.*

Theorem 3.1.2 ([4, Theorem 3.3]). *Let $I_2(\tilde{F})$ be a symmetrized skew tableau ideal. Then the initial ideal $in(I_2(\tilde{F}))$ is glicci.*

An immediate consequence of this by Theorem 2.1.6 is the following.

Corollary 3.1.3 ([4, Corollary 3.4]). *Symmetrized skew tableau ideals generated by 2×2 minors are Cohen-Macaulay.*

Recall that the question of whether every Cohen-Macaulay ideal is glicci is open. We will show in the following section that the answer is affirmative for skew tableau ideals generated by 2×2 minors. We will also need the following property of the ideals.

Proposition 3.1.4 ([4, Proposition 3.5]). *Symmetrized skew tableau ideals generated by 2×2 minors are prime.*

3.2 Skew Tableau Ideals Generated by 2×2 Minors are Glicci

In this section, we will always assume that row indices $b_i = a_i$, and so will use $T(a, c, d)$ to refer to $T(a, b, c, d)$. We will show that a symmetrized skew tableau ideal $I_2(\tilde{F})$ can be linked to a complete intersection. To do this, we will define skew tableaux G and H and an ideal $\mathfrak{c}$ generated by variables such that there is an elementary biliaison

$$\frac{I_2(\tilde{F})}{I_2(\tilde{H})} \cong \frac{I_2(G) + \mathfrak{c}}{I_2(\tilde{H})} (-1).$$

The variables generating $\mathfrak{c}$ will not appear in G, so we may continue the linkage starting from $I_2(G)$ alone using the same process as above (recall that unreflected skew tableau ideals are only a special case of reflected skew tableau ideals).

After we define the skew tableaux G and H and a complete intersection $\mathfrak{c}$, we will recall the height computation for the these ideals from [4] and show that the ideals $I_2(\tilde{H})$ and $I_2(G) + \mathfrak{c}$ have the appropriate heights for an elementary biliaison.

Definition 3.2.1. Let $F = T(a, c, d)$ be a skew tableau with n rows defined by the vectors $a, c, d \in \mathbb{Z}^k$, which represent k pairs of corners. We define a new tableau $G = T(a', c', d')$. Let q_i be the number of entries in row a_i and columns $n, n+1, \ldots, c_1 - 1$ of F, and note $q_1 = 0$. Define $a', c', d' \in \mathbb{Z}^k$ by:

$$\begin{aligned} a' &= (a_1 - 1, a_2, \ldots, a_k) \\ c' &= (d_1 - q_1, d_1 - q_2, \ldots, d_1 - q_k) \\ d' &= (d_1, \ldots, d_k) \end{aligned}$$

The skew tableau G contains the first $n-1$ rows of columns $n, n+1, \ldots, c_1 - 1$ of F along with the columns d_1 and higher. We preserve the labelling of the indeterminates in the first set of columns of G so that they match the indices of the original entries in F. Let $\mathfrak{c}$ be the ideal generated by all the other indeterminates in rows $1, \ldots, n-1$ of F. These are exactly the entries which are the upper left corner of a 2×2 submatrix of F with lower right corner at (a_1, d_1). They consist of all the indeterminates in columns 1 to $d_1 - 1$ and rows 1 to $n-1$ of F except for those in columns $n, n+1, \ldots, c_1 - 1$.

Example 3.2.2. Suppose F is a skew tableau with right corners $(a_1, d_1) = (5, 8)$ and $(a_2, d_2) = (2, 9)$ and first left corner $(a_1, c_1) = (5, 7)$. We have shown the symmetrized skew tableau $\tilde{F}$ corresponding to such an $\tilde{F}$ below. The dark gray entries are the generators of $\mathfrak{c}$, and the light gray entries form the new tableau G, with the labelling preserved from F.

$F =$

	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$	$x_{1,9}$
$x_{1,2}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$	
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$				$x_{4,7}$	$x_{4,8}$	
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$				$x_{5,7}$	$x_{5,8}$	

$G =$

$x_{1,5}$	$x_{1,6}$	$x_{1,8}$	$x_{1,9}$
$x_{2,5}$	$x_{2,6}$	$x_{2,8}$	$x_{2,9}$
$x_{3,5}$	$x_{3,6}$	$x_{3,8}$	
		$x_{4,8}$	

Definition 3.2.3. We define a tableau $H = T(a'', c'', d'')$ by removing the box (a_1, d_1) from F. There are two cases:

- **Case 1:** (a_1, d_1) is the only entry in row a_1 of F. Define $a'', c'', d'' \in \mathbb{Z}^k$ by:

$$\begin{aligned} a'' &= (a_1 - 1, a_2, \ldots, a_k) \\ c'' &= (c_1, \ldots, c_k) \\ d'' &= (d_1, \ldots, d_k) \end{aligned}$$

- **Case 2:** (a_1, d_1) is not the only entry in row a_1 of F. Define $a'', c'', d'' \in \mathbb{Z}^{k+1}$ by:

$$\begin{aligned} a'' &= (a_1, a_1 - 1, a_2, \ldots, a_k) \\ c'' &= (c_1, c_1, \ldots, c_k) \\ d'' &= (d_1 - 1, d_1, \ldots, d_k) \end{aligned}$$

Example 3.2.4. Suppose F is the same skew tableau as in the previous example. Then the new tableau H is formed by removing the box at $(5, 8)$.

$$F = \begin{array}{cccccccc} x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} \\ x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} \\ & & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & \\ & & & & & x_{4,7} & x_{4,8} & \\ & & & & & x_{5,7} & x_{5,8} & \end{array} \qquad H = \begin{array}{cccccccc} x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} \\ x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} \\ & & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & \\ & & & & & x_{4,7} & x_{4,8} & \\ & & & & & x_{5,7} & \square & \end{array}$$

This example corresponds to Case 2 in the definition of H. Case 1 is similar.

We cite the height computation from [4], *Theorem 3.3* for the initial ideals to show the formula for $ht\, I_2(F)$ and $ht\, I_2(\tilde{F})$. Note that the height is equal to the number of entries that occur as upper left corners of a 2×2 submatrix of F or $\tilde{F}$, respectively.

Proposition 3.2.5. *Let $\tilde{F}$ be a skew tableau with n rows defined by the pairs of corners $((a_i, c_i), (a_i, d_i))$ for $i = 1, \ldots, k$. We adopt the convention that $a_{k+1} = 1$. Then*

$$ht\, I_2(\tilde{F}) = \max\{0, n - c_k\} + \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - c_i)$$

and

$$ht\, I_2(F) = \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - c_i).$$

Proof. We will show the height formula for reflected skew tableau ideals, and then derive the formula for unreflected tableaux as a special case. First, assume that there is exactly one pair of corners for each row of F, so that $a_i = n + 1 - i$. In this case $a_i - a_{a+1} = (n + 1 - i) - (n + 1 - i - 1) = 1$, so our formula becomes

$$ht\, I_2(\tilde{F}) = \max\{0, n - c_k\} + \sum_{i=1}^{n-1} (d_i - c_i).$$

In [4], skew tableaux are defined using column indices λ_j and μ_j. Assuming we define our tableau F with a pair of corners on every row, the translation between notations is given by $\lambda_j = d_i$ and $\mu_j = c_i - 1$, where $j = n + 1 - i$. In the language of [4], our height formula becomes

$$ht\, I_2(\tilde{F}) = \max\{0, n - \mu_1 - 1\} + \sum_{j=2}^{n} (\lambda_j - \mu_j - 1),$$

which is exactly the formula proven in [4, Theorem 3.3].

Suppose now that our description of F requires fewer pairs of corners, i.e. that there are consecutive indices $i, \dots, i+m$ such that $c_i = c_{i+1} = \dots = c_{i+m}$ and $d_i = d_{i+1} = \dots = d_{i+m}$. Then in the height formula we may replace the summands from i to $i+m$ by the term $m \cdot (d_i - c_i) = (a_{i+m+1} - a_i) \cdot (d_i - c_i)$. After omitting these redundant pairs of corners and relabelling accordingly, this agrees with the claimed formula for reflected skew tableau ideals.

Now, recall from Remark 2.4.14 that an unreflected skew tableau ideal $I_2(F)$ may be regarded as a reflected skew tableau ideal $I_2(\tilde{F}')$, where F' is the tableau F shifted sufficiently far to the right so that reflecting across the diagonal does not introduce any new 2×2 submatrices. In particular, if F has n rows, we shift F so that $c_k > n$. In this case, $\max\{0, n - c_k\} = 0$, and the claimed formula for $ht\, I_2(F)$ is correct.

□

Corollary 3.2.6. *Let F, G, H, and $\mathfrak{c}$ be defined as above. Then*

$$ht\, I_2(\tilde{F}) = ht\, (I_2(G) + \mathfrak{c}) = ht\, I_2(\tilde{H}) + 1.$$

Proof. We compare the height of $(I_2(G) + \mathfrak{c})$ to the height of $I_2(\tilde{F})$. The variables in $\mathfrak{c}$ do not appear in any element of $I_2(G)$, so $ht\, (I_2(G) + \mathfrak{c}) = ht\, I_2(G) + ht\, \mathfrak{c}$. First, assume that $q_i = 0$ for all i, which occurs if $c_1 = n$. The proposition above gives that

$$\begin{aligned} ht\, I_2(G) &= \sum_{i=1}^{k} (a'_i - a'_{i+1})(d'_i - c'_i) \\ &= \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - d_1) \end{aligned}$$

Note that the first summand is zero, so we may ignore the -1 in a'_1. The ideal $\mathfrak{c}$ is generated by all variables in the columns of F up to column $d_1 - 1$, so from [4], *Theorem 3.3* we have that

$$ht\, \mathfrak{c} = \max\{0, n - c_k\} + \sum_{i=1}^{k} (a_i - a_{i+1})(d_1 - c_i).$$

Putting these calculations together, we obtain that

$$\begin{aligned} &ht\, I_2(G) + ht\, \mathfrak{c}. \\ &= \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - d_1) + \max\{0, n - c_k\} + \sum_{i=1}^{k} (a_i - a_{i+1})(d_1 - c_i) \\ &= \max\{0, n - c_k\} + \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - d_1) + \sum_{i=1}^{k} (a_i - a_{i+1})(d_1 - c_i) \\ &= \max\{0, n - c_k\} + \sum_{i=1}^{k} (a_i - a_{i+1})(d_i - c_i) \\ &= ht\, I_2(\tilde{F}). \end{aligned}$$

If some of the q_i are nonzero, then we have

$$\begin{aligned} ht\, I_2(G) &= \sum_{i=1}^{k}(a_i - a_{i+1})(d_i - d_1 + q_i) \\ &= \sum_{i=1}^{k}(a_i - a_{i+1})(d_i - d_1) + \sum_{i=1}^{k}(a_i - a_{i+1})(q_i) \end{aligned}$$

This is the same computation as above, except that we have added the second summation because of the q_i. But this second summation exactly counts the number of variables excluded from $\mathfrak{c}$, so by the computations above and again by [4, Theorem 3.3], we get that

$$\begin{aligned} ht\, I_2(G) + ht\, \mathfrak{c} &= ht\, I_2(\tilde{F}) + \sum_{i=1}^{k}(a_i - a_{i+1})(q_i) - \sum_{i=1}^{k}(a_i - a_{i+1})(q_i) \\ &= ht\, I_2(\tilde{F}). \end{aligned}$$

To compute the height of $I_2(\tilde{H})$, we must consider the situation in which there is only one entry in the last row of F separately from the situation in which there are more than one.

- **Case 1:** (a_1, c_1) is the last entry in the n^{th} row of F.

  In the first pair of corners, $c_1 = d_1$ and H has $n-1$ rows, so the formula is

$$\begin{aligned} ht\, I_2(\tilde{H}) &= \max\{0, n-1-c_k\} + (a_1 - 1 - a_2)(d_1 - c_1) + \sum_{i=2}^{k-1}(a_i - a_{i+1})(d_i - c_i) \\ &= \max\{0, n-1-c_k\} + \sum_{i=2}^{k-1}(a_i - a_{i+1})(d_i - c_i) \end{aligned}$$

  Since $c_1 = d_1 = n$ and $c_k < c_1$ (as long as $I_2(\tilde{F})$ is nonempty), it follows that $\max\{0, n-1-c_k\} = n-1-c_k$. Therefore

$$\begin{aligned} ht\, I_2(\tilde{H}) &= (n-1-c_k) + \sum_{i=2}^{k-1}(a_i - a_{i+1})(d_i - c_i) \\ &= ht\, I_2(\tilde{F}) - 1. \end{aligned}$$

- **Case 2:** (a_1, c_1) is not the last entry in the n^{th} row of F.

Now $\tilde{H}$ is defined by an extra pair of corners, and so the formula is

$$\begin{aligned}
ht\, I_2(\tilde{H}) =& \max\{0, n-c_k\} + (a_1-(a_1-1))(d_1-1-c_1) + (a_1-1-a_2)(d_1-c_1) \\
&+ \sum_{i=2}^{k-1}(a_i-a_{i+1})(d_i-c_i) \\
=& \max\{0, n-c_k\} + (d_1-1-c_1) + (a_1-1-a_2)(d_1-c_1) \\
&+ \sum_{i=2}^{k-1}(a_i-a_{i+1})(d_i-c_i) \\
=& \max\{0, n-c_k\} - 1 + (a_1-a_2)(d_1-c_1) + \sum_{i=2}^{k-1}(a_i-a_{i+1})(d_i-c_i) \\
=& \max\{0, n-c_k\} - 1 + \sum_{i=1}^{k-1}(a_i-a_{i+1})(d_i-c_i) \\
=& ht\, I_2(\tilde{F}) - 1.
\end{aligned}$$

Thus in both cases, $ht\, I_2(\tilde{F}) = ht\,(I_2(G)+\mathfrak{c}) = ht\, I_2(\tilde{H}) + 1$.

□

Liaison classification of skew tableau ideals generated by 2×2 minors

We now prove the main result of this chapter:

Theorem 3.2.7. *Symmetrized skew tableau ideals generated by 2×2 minors are glicci.*

Proof. We will show how to link an arbitrary ideal $I_2(\tilde{F})$ to an ideal generated by a list of variables. Using the tableaux G and H and the ideal $\mathfrak{c}$ defined in Definition 3.2.1 and Definition 3.2.3, we will show an elementary biliaison:

$$\frac{I_2(\tilde{F})}{I_2(\tilde{H})} \cong \frac{I_2(G)+\mathfrak{c}}{I_2(\tilde{H})}\ (-1).$$

We will choose homogeneous polynomials f and g of order 1 and 2, respectively, such that

$$f \cdot I_2(\tilde{F}) + I_2(\tilde{H}) = g \cdot (I_2(G)+\mathfrak{c}) + I_2(\tilde{H}),$$

and both $I_2(\tilde{H}) : f = I_2(\tilde{H})$ and $I_2(\tilde{H}) : g = I_2(\tilde{H})$. The equality may be shown at the level of generators. Any generator $a \in I_2(\tilde{F})$ corresponds to a generator $b \in I_2(G)+\mathfrak{c}$ such that

$$f \cdot a - g \cdot b \in I_2(\tilde{H}). \qquad (\star)$$

Conversely, the same argument will guarantee that for any $b \in I_2(G)+\mathfrak{c}$ there is a corresponding $a \in I_2(\tilde{F})$ that gives the same equality.

From Definition 3.2.3, it is clear that if a minor $a \in I_2(\tilde{F})$ does not contain the variable at location (a_1, d_1), then $a \in I_2(\tilde{H})$. Similarly, from Definition 3.2.1, if b is a generating minor of $I_2(G)$, then $b \in I_2(\tilde{H})$. So on the left hand side, we only need to consider generators $a \in I_2(\tilde{F})$ which contain the variable at location (a_1, d_1), and on the right hand side from $I_2(G) + \mathfrak{c}$ we only need to consider variables that generate $\mathfrak{c}$. Let $a \in I_2(\tilde{F})$ come from a 2×2 submatrix of $\tilde{F}$ with lower right corner at (a_1, d_1). Then the variable b which is the upper left corner of this submatrix is a generator of $\mathfrak{c}$. We need to show the equality $(\star)$ for each of these pairs a and b with the fixed f and g that we have chosen.

To avoid unnecessary indices, we will use (n, p) for the location of the right corner (a_1, d_1). Let $a \in I_2(\tilde{F})$ be a minor with upper left corner at location (i, j) and lower right corner at location (n, p). This will fall into one of the following three types, depending on how much of the minor comes from the unreflected part of $\tilde{F}$:

- Type 1: $(i, j) \in F$ and $(n, j) \in F$. Then $a = \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{n,j} & x_{n,p} \end{vmatrix}$ and $b = x_{i,j}$.
- Type 2: $(i, j) \in F$ and $(n, j) \notin F$. Then $a = \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$ and $b = x_{i,j}$.
- Type 3: $(i, j) \notin F$ and $(n, j) \notin F$. Then $a = \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$ and $b = x_{j,i}$.

We must consider the case where $(n, p) = (n, n)$ separately from the case where $n < p$. We will show the equality $(\star)$ for all three types of minors in both cases. Each argument is accompanied by an illustrative example with the part of the tableau below the diagonal shaded in gray. We will at times ignore the last few rows of a tableau, in accordance with Remark 2.4.12, in order to avoid excessively large diagrams.

We choose g to be the 2×2 minor from entries at locations

$$\begin{matrix} (1,k) & (1,p) \\ (n,k) & (n,p) \end{matrix}$$

and f to be the entry at $(1, k)$, where k is the farthest right index such that all four locations are in $\tilde{F}$. Note that $\deg(f) - \deg(g) = -1$.

Case 1: $x_{n,p}$ is not the only entry of row n of F.

In Case 1, $g = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{n,k} & x_{n,p} \end{vmatrix}$ and $f = x_{1,k}$. Since $x_{n,p}$ is not the only entry of row n in F, $k = p - 1$.

Type 1: $(i, j) \in F$ and $(n, j) \in F$.

A generator of $I_2(\tilde{F})$ of Type 1 has the form

$$a = \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{n,j} & x_{n,p} \end{vmatrix}$$

and corresponds to a generator $b = x_{i,j}$ in $\mathfrak{c}$. We will show $f \cdot a = g \cdot b + h$ for some $h \in I_2(\tilde{H})$, i.e. that

$$x_{1,k} \cdot \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{n,j} & x_{n,p} \end{vmatrix} = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{n,k} & x_{n,p} \end{vmatrix} \cdot x_{i,j} + h.$$

This is true if we choose

$$h = x_{1,p} \cdot \begin{vmatrix} x_{i,j} & x_{i,k} \\ x_{n,j} & x_{n,k} \end{vmatrix} - x_{n,j} \cdot \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{i,k} & x_{i,p} \end{vmatrix},$$

which uses the following entries:

					$x_{1,k}$	$x_{1,p}$
				$x_{i,j}$	$x_{i,k}$	$x_{i,p}$
				$x_{n,j}$	$x_{n.k}$	$x_{n,p}$

Note that $x_{i,k} \in I_2(\tilde{H})$, since $x_{1,k}, x_{n,k} \in H$ and $1 \leq i < n$. Thus for any choice of Type 1 minor, $h \in I_2(\tilde{H})$.

Type 2: $(i,j) \in F$ and $(n,j) \notin F$.

A generator of $I_2(\tilde{F})$ of Type 2 has the form

$$a = \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$$

and corresponds to a generator $b = x_{i,j}$ in $\mathfrak{c}$. We will show $f \cdot a = g \cdot b + h$ for some $h \in I_2(\tilde{H})$, i.e. that

$$x_{1,k} \cdot \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix} = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{n,k} & x_{n,p} \end{vmatrix} \cdot x_{i,j} + h.$$

This is true if we choose

$$h = x_{1,p} \cdot \begin{vmatrix} x_{i,j} & x_{i,k} \\ x_{j,n} & x_{n,k} \end{vmatrix} - x_{j,n} \cdot \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{i,k} & x_{i,p} \end{vmatrix},$$

which uses the following entries:

					$x_{1,k}$	$x_{1,p}$
			$x_{i,j}$		$x_{i,k}$	$x_{i,p}$
			$x_{j,n}$		$x_{n.k}$	$x_{n,p}$

We require that $x_{i,k} \in \tilde{H}$, and this is true since $k = p - 1$, $x_{1,k} \in H$ and $x_{n,k} \in H$, and $1 \leq i < n$. Thus for any elements a and b of Type 2, there exists an appropriate $h \in I_2(\tilde{H})$.

Type 3: $(i, j) \notin F$ and $(n, j) \notin F$.

A generator of $I_2(\tilde{F})$ of Type 3 has the form

$$a = \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$$

and corresponds to a generator $b = x_{j,i}$ in $\mathfrak{c}$. We will show $f \cdot a = g \cdot b + h$ for some $h \in I_2(\tilde{H})$, i.e. that

$$x_{1,k} \cdot \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix} = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{n,k} & x_{n,p} \end{vmatrix} \cdot x_{j,i} + h.$$

This is true if we choose

$$h = x_{1,p} \cdot \begin{vmatrix} x_{j,i} & x_{i,k} \\ x_{j,n} & x_{n,k} \end{vmatrix} - x_{j,n} \cdot \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{i,k} & x_{i,p} \end{vmatrix},$$

which uses the following entries:

					$x_{1,k}$	$x_{1,p}$
$x_{j,i}$					$x_{i,k}$	$x_{i,p}$
$x_{j,n}$					$x_{n.k}$	$x_{n,p}$

We require that $x_{i,k} \in \tilde{H}$, and this is true since $x_{1,k}, x_{n,k} \in H$ and $1 \leq i < n$. Thus for any elements a and b of Type 3, there exists an appropriate $h \in I_2(\tilde{H})$.

Case 2: $x_{n,p}$ is the only entry of row n of $\tilde{F}$, so $k < n$

In Case 2, $g = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{k,n} & x_{n,p} \end{vmatrix}$ and $f = x_{1,k}$. Note that Type 1 pairs do not occur in Case 2.

Type 2: $(i,j) \in F$ and $(n,j) \notin F$.

A generator of $I_2(\tilde{F})$ of Type 2 has the form

$$a = \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$$

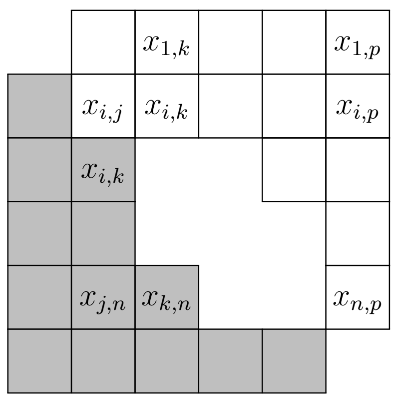
and corresponds to a generator $b = x_{i,j}$ in $\mathfrak{c}$. We will show $f \cdot a = g \cdot b + h$ for some $h \in I_2(\tilde{H})$, i.e. that

$$x_{1,k} \cdot \begin{vmatrix} x_{i,j} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix} = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{k,n} & x_{n,p} \end{vmatrix} \cdot x_{i,j} + h.$$

This is true if we take

$$h = x_{1,p} \cdot \begin{vmatrix} x_{i,j} & x_{i,k} \\ x_{j,n} & x_{k,n} \end{vmatrix} - x_{j,n} \cdot \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{i,k} & x_{i,p} \end{vmatrix},$$

which uses the following entries:



We need $x_{i,k} \in H$. Since $x_{i,j} \in H$, it is also true that $x_{1,j} \in H$. By definition of k, $j \leq k < p$, and since and both $x_{i,j} \in H$ and $x_{i,p} \in H$, it follows that $x_{i,k} \in H$. Thus for any elements a and b of Type 2, there exists an appropriate $h \in I_2(\tilde{H})$.

Type 3: $(i,j) \notin F$ and $(n,j) \notin F$.

A generator of $I_2(\tilde{F})$ of Type 2 has the form

$$a = \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix}$$

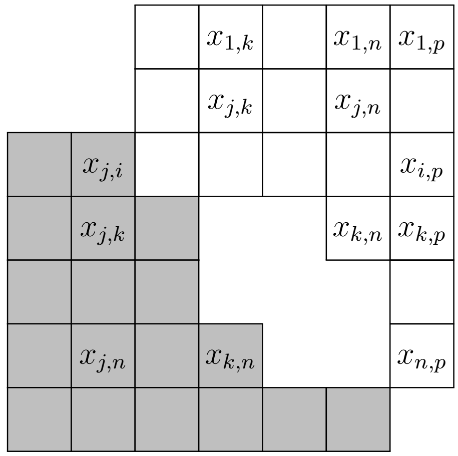
and corresponds to a generator $b = x_{j,i}$ in $\mathfrak{c}$. We will show $f \cdot a = g \cdot b + h$ for some $h \in I_2(\tilde{H})$, i.e. that

$$x_{1,k} \cdot \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,n} & x_{n,p} \end{vmatrix} = \begin{vmatrix} x_{1,k} & x_{1,p} \\ x_{k,n} & x_{n,p} \end{vmatrix} \cdot x_{j,i} + h.$$

This is true if we take

$$h = x_{1,n} \cdot \begin{vmatrix} x_{j,i} & x_{i,p} \\ x_{j,k} & x_{k,p} \end{vmatrix} - x_{j,i} \cdot \begin{vmatrix} x_{1,n} & x_{1,p} \\ x_{k,n} & x_{k,p} \end{vmatrix} - x_{i,p} \cdot \begin{vmatrix} x_{1,k} & x_{1,n} \\ x_{j,k} & x_{j,n} \end{vmatrix},$$

which uses the following entries:



We need $x_{1,n}, x_{j,k}, x_{k,p} \in H$. Since $x_{k,n} \in H$, it is also true that $x_{1,n} \in H$. Note that since $x_{j,i} \in H$, we also know that $x_{1,i} \in H$, so $i \leq k < n$ by the definition of k. Since also $x_{j,i}$ and $x_{j,n}$ are both in H, it follows that $x_{j,k} \in H$. Finally, since $k < n$ and $x_{n,p} \in H$, it follows that $x_{k,p} \in H$. Thus for any corresponding pair of elements a and b of Type 3, there exists an appropriate $h \in I_2(\tilde{H})$.

We have shown that for every choice of generator $a \in I_2(\tilde{F})$, there is a generator $b \in I_2(G) + \mathfrak{c}$ such that

$$f \cdot a - g \cdot b \in I_2(\tilde{H}).$$

Given a variable $b \in \mathfrak{c}$, we take a to be any 2×2 minor from a submatrix with lower corner at x_{a_1,d_1} and upper left corner b. Then the same argument as above works in the opposite direction to show the same conclusion. Thus we have determined that for our chosen f and g,

$$f \cdot I_2(\tilde{F}) + I_2(\tilde{H}) = g \cdot (I_2(G) + \mathfrak{c}) + I_2(\tilde{H}).$$

By Corollary 3.2.6, $ht\, I_2(\tilde{F}) = ht\, (I_2(G)+\mathfrak{c}) = ht\, I_2(\tilde{H})+1$, and by Corollary 3.1.3 and Proposition 3.1.4 from [4], ideals of the form $I_2(\tilde{F})$ are Cohen-Macaulay and prime. Note that the minor g involves the variable x_{a_1,d_1}, which is not present in $I_2(\tilde{H})$, so $g \notin I_2(\tilde{H})$, and also that f is just a variable, so $f \notin I_2(\tilde{H})$. Thus f and g are nonzero divisors modulo $I_2(\tilde{H})$ that define an isomorphism

$$\frac{I_2(\tilde{F})}{I_2(\tilde{H})} \cong \frac{I_2(G) + \mathfrak{c}}{I_2(\tilde{H})}\,(-1).$$

By the above comments, this is an elementary biliaison between $I_2(\tilde{F})$ and $I_2(G)+\mathfrak{c}$, and so by Theorem 2.1.5, the two ideals may be G-linked in two steps.

The variables that generate $\mathfrak{c}$ are not present in the tableau G, so we may continue the linkage from the ideal $I_2(G)$ alone. Recall from Remark 2.4.14 that unsymmetrized ideals are a special case of symmetrized ideals, so we may define another tableau G_2 and an ideal $\mathfrak{c}_2$ such that $I_2(G)$ is linked to $I_2(G_2)+\mathfrak{c}_2$ in two steps by the same result. The tableau G_i is strictly smaller than the tableau G_{i-1}, so this process may be repeated a finite number of times N until the ideal $I_2(G_N)$ is empty. This gives us a series of links from $I_2(\tilde{F})$ to $\mathfrak{c}+\mathfrak{c}_2+\cdots+\mathfrak{c}_N$, thus showing that reflected skew tableau ideals generated by 2×2 minors are glicci.

□

The choice of the element h for Type 3 minors in Case 2 of the preceding proof has three terms instead of two, which is different from the form of the expression in Proposition 2.3.10. This happens because that formula would involve the entry at location (i,k), and this need not be present in a skew tableau.

Chapter 4 Skew Tableau Ideals Generated by $t \times t$ Minors

For $t \geq 3$, skew tableau ideals from $t \times t$ minors lose some of the nice properties enjoyed by the other classes of determinantal ideals that we have mentioned. In particular, the set of minors generating the ideal do not necessarily form a Gröbner basis, and the ideals need not be prime. We give examples of ideals where these properties fail. On the other hand, there are special cases for which the ideals enjoy some of these nice properties, but for which the ideals are equivalent to other known classes of ideals. We discuss one such example in section 4.1, where we give an original proof of the fact that skew tableau ideals from tableaux with no holes are glicci. Unsymmetrized skew tableau ideals are a distinct class of ideals from those studied previously, and for these we have shown that the set of minors forms a Gröbner basis. The proof of this fact involves a somewhat surprising application of Theorem 2.3.9 that describes the output of the division algorithm. We close the chapter with some plans for further study.

4.1 Skew tableaux with no holes

Definition 4.1.1. A *skew tableau with no holes* is one which has a left corner at every location on the main diagonal and no left corners off it. In such a skew tableau $F = T(a, b, c, d)$ with n rows, we will set $b = (1, 2, \ldots, n)$ and $c = (1, 2, \ldots, n)$. Since the left corners are already determined, it is enough for us to specify a list of right corners (a_i, d_i) and consider $F = T(a, d)$.

Our goal is to show that skew tableau ideals with no holes can be G-linked to a complete intersection. From a given skew tableau ideal $I_t(\tilde{F})$, we will define new skew tableaux $\tilde{G}$ and $\tilde{H}$, and then define an elementary biliaison

$$I_t(\tilde{F})/I_\tau(\tilde{H}) \cong I_{t'}(\tilde{G})/I_\tau(\tilde{H})\ (-1).$$

This will determine a pair of links from $I_t(\tilde{F})$ to $I_{t'}(\tilde{G})$. The vector t' will agree with t except in a single entry, at which place it will be one less. Thus we may repeat this process a finite number of times to link to a skew tableau ideal with vector $t' = (1, \ldots, 1)$, which defines a complete intersection. Since skew tableau ideals are a special case of symmetrized skew tableau ideals (see Remark 2.4.14), we will only treat the symmetrized case in this section.

Remark 4.1.2. Ideals from skew tableaux with no holes are equivalent to ideals generated by minors of mixed sixed in a symmetric ladder of a symmetric matrix. Although a Gröbner basis calculation was given for these ideals in [12], and it was shown in [9] that ideals in this class are glicci. We provide an independent proof of both results and note some small omissions in the second proof. Our approach depends on the determinantal identity we have shown in Proposition 2.3.10, and does not depend on the unproven Lemma 2.3.6. For our purposes, these new proofs also have the advantage that the arguments could generalize more easily to ideals generated in arbitrary skew tableaux.

First, we compute a Gröbner basis for a skew tableau ideal with no holes, and use this to show that the ideals are prime and to calculate the their heights. Our Gröbner basis calculation will refer to *partially-symmetric matrices*. These are matrices of the form

$$(S \,|\, X)$$

in which S is a generic symmetric matrix and X is a generic matrix of indeterminates. For the following lemma, we specify a diagonal term order as defined in Definition 2.2.1.

Lemma 4.1.3 (c.f. [12, Theorem 4.7])**.** *Let F be a skew tableau with no holes. For any diagonal term order, the minors generating $I_t(\tilde{F})$ form a Gröbner basis.*

Proof. Suppose $\tilde{F}$ is defined by the subregions $\tilde{F}_i$ for $i = 1, \dots, k$ as in Definition 2.4.6. Recall from Remark 2.4.12 that if A is any $t_i \times t_i$ submatrix of $\tilde{F}_i$, then either $A \in \tilde{F}_i \cap X_i$ or $A^{\mathrm{tr}} \in \tilde{F}_i \cap X_i$. So $I_{t_i}(\tilde{F}_i) = I_{t_i}(\tilde{F}_i \cap X_i)$. We will use this fact repeatedly. Let $\mathfrak{G}_i$ be the set of generating minors in $I_{t_i}(\tilde{F}_i)$. We are trying to show that $\mathfrak{G} = \bigcup_{i=1}^{k} \mathfrak{G}_i$ is a Gröbner basis of $I_t(\tilde{F}) = \sum_{i=1}^{k} I_{t_i}(\tilde{F}_i) = \sum_{i=1}^{k} I_{t_i}(\tilde{F}_i \cap X_i)$.

We argue by induction on k, the number of right corners that define F. If $k = 1$, then $F_1 \cap X_1$ is a partially-symmetric matrix, and so $\mathfrak{G}$ is a Gröbner basis of $I_t(\tilde{F}_1 \cap X_1) = I_t(\tilde{F})$ by [2, Remark 4.1.6 and Theorem 4.3.2].

If $k = 2$, then $\tilde{F} = \tilde{F}_1 \bigcup \tilde{F}_2$ is defined by the two right corners (a_1, d_1) and (a_2, d_2), and $(\tilde{F}_1 \cap X_1) \bigcup (\tilde{F}_2 \cap X_2)$ is contained in a partially-symmetric matrix X of size $a_1 \times d_2$. Then $I_{t_1}(\tilde{F}_1 \cap X_1) + I_{t_2}(\tilde{F}_2 \cap X_2)$ is a symmetric ladder ideal by [2, Section 4.6], and so $\mathfrak{H}$, the set of generating minors, forms a Gröbner basis of the ideal by [2, Theorem 4.4.4]. Clearly a skew tableau $\tilde{F}$ satisfies the conditions of Lemma 2.2.5, so $\mathfrak{G} = \mathfrak{H} \cap K[F]$ is a Gröbner basis of $I_t(\tilde{F})$.

Assume the claim is true for $k - 1$. Let $\tilde{F}' = \bigcup_{i=1}^{k-1} \tilde{F}_i$, and let $t' = (t_1, \dots, t_{k-1})$. Then

$$\tilde{F} = \tilde{F}' \cup \tilde{F}_k \quad \text{and} \quad I_t(\tilde{F}) = I_{t'}(\tilde{F}') + I_{t_k}(\tilde{F}_k).$$

Let $\mathfrak{G}' = \bigcup_{i=1}^{k-1} \mathfrak{G}_i$. Then $\mathfrak{G}'$ is a Gröbner basis of $I_{t'}(\tilde{F}')$ and $\mathfrak{G}_k$ is a Gröbner basis of $I_{t_k}(\tilde{F}_k)$ by the induction hypothesis, and $\mathfrak{G} = \mathfrak{G}' \bigcup \mathfrak{G}_k$. Let $f \in \mathfrak{G}'$ and $g \in \mathfrak{G}_k$. Note that $f \in \mathfrak{G}_j$ for some $1 \leq j \leq k - 1$. By the argument above when $k = 2$, $\mathfrak{G}_j \bigcup \mathfrak{G}_k$ is a Gröbner basis of $I_{t_j}(\tilde{F}_j) + I_{t_k}(\tilde{F}_k)$. Then by Lemma 2.2.6, there is an element

$$h \in I_{t_j}(\tilde{F}_j) \cap I_{t_k}(\tilde{F}_k)$$

such that

$$in(h) = \mathrm{lcm}(in(f), in(g)).$$

But this means that

$$h \in I_{t_j}(\tilde{F}_j) \cap I_{t_k}(\tilde{F}_k) \subset I_{t'}(\tilde{F}') \cap I_{t_k}(\tilde{F}_k),$$

so $\mathfrak{G}' \bigcup \mathfrak{G}_k$ is a Gröbner basis of $I_t(\tilde{F}) = I_{t'}(\tilde{F}') + I_{t_k}(\tilde{F}_k)$, again by Lemma 2.2.6. $\square$

Corollary 4.1.4. *Let $I_t(\tilde{F})$ be as above. Then the initial ideal $in(I_t(\tilde{F}))$ is squarefree.*

Since skew tableau ideals from tableaux with no holes are a special case of symmetric ladder determinantal ideals, we immediately get the following property.

Proposition 4.1.5 (c.f. [9, Proposition 1.7])**.** *Symmetrized skew tableau ideals from skew tableaux with no holes are prime.*

We now describe how to define the skew tableau ideals $I_{t'}(\tilde{G})$ and $I_\tau(\tilde{H})$ from an arbitrary skew tableau ideal $I_t(\tilde{F})$, from which we will produce an elementary biliaison.

Definition 4.1.6. Let $F = T(a, d)$ be a skew tableau with no holes, with $t = (t_1, \ldots, t_k)$ its corresponding vector. Suppose F has n rows. Choose any r such that $t_r > 1$, and such that (a_r, d_r) is the farthest right corner in row a_r and the lowest corner in column d_r. Define the tableau $G = T(a', d')$ by the vectors $a', d' \in \mathbb{Z}^k$, where

$$\begin{aligned} a' &= (a_1, \ldots, a_{r-1}, a_r - 1, a_{r+1}, \ldots, a_k) \\ d' &= (d_1, \ldots, d_{r-1}, d_r - 1, d_{r+1}, \ldots, d_k) \end{aligned}$$

and let $t' = (t_1, \ldots, t_{r-1}, t_r - 1, t_{r+1}, \ldots, t_k) \in \mathbb{Z}^k$.

Example 4.1.7. Suppose F is the tableau with right corners $(5, 7)$ and $(3, 10)$ and vector $t = (2, 2)$. We choose $r = 2$, which corresponds to the corner $(3, 8)$. Then G has right corners $(5, 7)$ and $(2, 9)$ and vector $t' = (2, 1)$. The relevant parts of F and G are shown below.

$$F = \begin{array}{|c|c|c|c|c|c|c|} \hline \cdots & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} & x_{1,10} \\ \hline \cdots & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} & x_{2,10} \\ \hline \cdots & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & x_{3,9} & x_{3,10} \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & x_{4,6} & x_{4,7} & \multicolumn{3}{c}{} \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & x_{5,6} & x_{5,7} & \multicolumn{3}{c}{} \\ \cline{3-4} \end{array} \qquad G = \begin{array}{|c|c|c|c|c|c|c|} \hline \cdots & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} & \\ \hline \cdots & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} & \\ \hline \cdots & x_{3,5} & x_{3,6} & x_{3,7} & & & \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & x_{4,6} & x_{4,7} & \multicolumn{3}{c}{} \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & x_{5,6} & x_{5,7} & \multicolumn{3}{c}{} \\ \cline{3-4} \end{array}$$

Definition 4.1.8. Let $H = T(a'', d'')$ be the skew tableau obtained by removing the location (a_r, d_r) from F. There are two cases:

- **Case 1:** Suppose $r = 1$ and $d_1 = n$, so there is only one entry in the last row of F. Let H be defined by the vectors $a''', d'' \in \mathbb{Z}^k$, where

$$\begin{aligned} a'' &= (a_1 - 1, a_2, \ldots, a_k) \\ d'' &= (d_1, \ldots, d_k). \end{aligned}$$

  Define the corresponding vector to be $\tau = t$.

- **Case 2:** Suppose that either $r > 1$ or that $r = 1$ and $d_1 > c_1$. Let H be defined by the vectors $a''', d'' \in \mathbb{Z}^{k+1}$, where

$$\begin{aligned} a'' &= (a_1, \ldots, a_{r-1}, a_r, a_r - 1, a_{r+1}, \ldots, a_k) \\ d'' &= (d_1, \ldots, d_{r-1}, d_r - 1, d_r, d_{r+1}, \ldots, d_k) \end{aligned}$$

  Define the corresponding vector to be $\tau = (t_1, \ldots, t_r, t_r, \ldots, t_k) \in \mathbb{Z}^{k+1}$, which is the same as t but with the r^{th} entry repeated.

Example 4.1.9. We consider the same example as the previous one. Given the same skew tableau F, we define the skew tableau H according to the description in Case 2, with right corners $(5, 7)$, $(3, 9)$, and $(2, 10)$ and vector $\tau = (2, 2, 2)$. The relevant parts of F and H are shown below.

$F =$

$\cdots$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$	$x_{1,9}$	$x_{1,10}$
$\cdots$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$
$\cdots$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$	$x_{3,9}$	$x_{3,10}$
		$x_{4,6}$	$x_{4,7}$			
		$x_{5,6}$	$x_{5,7}$			

$H =$

$\cdots$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$	$x_{1,9}$	$x_{1,10}$
$\cdots$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$
$\cdots$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$	$x_{3,9}$	
		$x_{4,6}$	$x_{4,7}$			
		$x_{5,6}$	$x_{5,7}$			

We will calculate the height of a skew tableau ideal by specifying a *target region* in F, which reflects the fact that we will eventually link to the ideal generated by the variables in that region. The number of distinct variables in the target region is the height of the corresponding skew tableau ideal.

Definition 4.1.10. Let F be a skew tableau defined by k pairs of corners and with associated vector t. We adopt the convention that $t_0 = t_1$ and $a_0 = d_0 = a_1$, which may be thought of as adding a redundant 0^{th} right corner at the lowest entry on the main diagonal of F. Define the following ideals:

- $\mathfrak{f}_0 = \left\langle x_{i,j} \,|\, i \leq j, 1 \leq i \leq a_0 - t_0 + 1 \text{ and } 1 \leq j \leq d_0 - t_0 + 1 \right\rangle$
- for $r \geq 1$, $\mathfrak{f}_r = \left\langle x_{i,j} \,|\, 1 \leq i \leq a_r - t_r + 1 \text{ and } d_{r-1} - t_{r-1} + 2 \leq j \leq d_r - t_r + 1 \right\rangle$.

Let $\tilde{T}$ be the subset of indeterminates in F which generate the $\mathfrak{f}_i$'s. We will refer to $\tilde{T}$ as the *target region*, and the ideal $\tilde{\mathfrak{f}} = \mathfrak{f}_0 + \cdots + \mathfrak{f}_k$ as the *target ideal*.

Example 4.1.11. Let $\tilde{F}$ be the symmetrized skew tableau with right corners at $(a_1, d_1) = (5, 7)$ and $(a_2, d_2) = (3, 10)$ and vector $t = (3, 2)$. In this example, $a_0 = d_0 = 5$. The first five rows of $\tilde{F}$ are shown below, with the target region $\tilde{T}$ shaded in gray. The entries corresponding to $\mathfrak{f}_0$ are dark gray and the entries corresponding to $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are light gray.

$$F = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} & x_{1,10} \\
\hline
x_{1,2} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} & x_{2,10} \\
\hline
x_{1,3} & x_{2,3} & x_{3,3} & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & x_{3,9} & x_{3,10} \\
\hline
x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} & x_{4,5} & x_{4,6} & x_{4,7} \\
\cline{1-7}
x_{1,5} & x_{2,5} & x_{3,5} & x_{4,5} & x_{5,5} & x_{5,6} & x_{5,7} \\
\cline{1-7}
\end{array}$$

Here $\mathfrak{f}_1 = \langle x_{i,j} \,|\, 1 \leq i \leq 3 \,\text{and}\, 4 \leq j \leq 5 \rangle$ and $\mathfrak{f}_2 = \langle x_{i,j} \,|\, 1 \leq i \leq 2 \,\text{and}\, 6 \leq j \leq 9 \rangle$.

Proposition 4.1.12. *Let $t_0 = t_1$ and $a_0 = c_0 = d_0 = a_1$, as in the previous definition. Define $q_0 = \binom{a_0 - t_0 + 2}{2}$, and for $r \geq 1$, let $q_r = \max\{0, (a_r - t_r + 1)(d_r - t_r - d_{r-1} + t_{r-1})\}$. Then*

$$ht\, I_t(\tilde{F}) = q_0 + q_1 + \cdots + q_r\,.$$

Proof. First, we note that $|\tilde{T}| = q_0 + q_1 + \cdots + q_r$. In particular, $\mathfrak{f}_i$ has q_i generators, and the generating sets of the $\mathfrak{f}_i$'s are disjoint. The variables generating $\mathfrak{f}_0$ come from a symmetric submatrix of $\tilde{F}$ of size $(a_0 - t_0 + 1) \times (d_0 - t_0 + 1)$, and since $a_0 = d_0$, this is generated by

$$q_0 = (a_0 - t_0 + 1) \cdot (d_0 - t_0 + 1) = \binom{a_0 - t_0 + 2}{2}$$

distinct variables. For $r \geq 1$, it follows from Lemma 2.4.13 that the variables generating $\mathfrak{f}_r$ come from a rectangular submatrix of $\tilde{F}$ which (if it is nonempty) has size $(a_r - t_r + 1) \times (d_r - t_r - d_{r-1} + t_{r-1})$, and this region is completely contained in the part of $\tilde{F}$ above the diagonal, so that all q_r variables in it are distinct. Thus $\mathfrak{f}_r$ is generated by

$$q_r = \max\{0, (a_r - t_r + 1)(d_r - t_r - d_{r-1} + t_{r-1})\}$$

distinct variables.

Choose any diagonal term order. By Lemma 4.1.3, the set of minors generating $I_t(\tilde{F})$ form a Gröbner basis. Using the fact that $ht\, in(I_t(\tilde{F})) = ht\, I_t(\tilde{F})$, we only need to show that $ht\, in(I_t(\tilde{F})) = |\tilde{T}|$.

First, we observe that $ht\, in(I_t(\tilde{F}))$ is equal to the minimal height of an associated prime of $I_t(\tilde{F})$, which is the minimal cardinality among subsets S of indeterminates in $\tilde{F}$ with the property that each monomial in a set of generators of $in(I_t(\tilde{F}))$ contains a variable from S. Now, it is clear that every monomial in a set of generators of $in(I_t(\tilde{F}))$ contains a variable from the target region $\tilde{T}$, since the upper left corner from every minor generating $I_t(\tilde{F})$ is an element of $\tilde{T}$. We claim that no smaller cardinality subset of indeterminates of $\tilde{F}$ satisfies this condition.

Suppose there is a subset S such that each monomial in a set of generators of $in(I_t(\tilde{F}))$ contains a variable from S, but that $|S| < |\tilde{T}|$. For each integer u that makes the set nonempty, define δ_u to be the set of variables that are u spaces off the main diagonal of $\tilde{F}$, i.e.

$$\delta_u = \{x_{i,j} \in \tilde{F} \,\text{at a location of the form}\, (a, a+u)\}.$$

Note that every variable along one of these diagonals is distinct. Now, since $|S| < |\tilde{T}|$, for some u it must be true that $|S \cap \delta_u| < |\tilde{T} \cap \delta_u|$. The farthest down entry in $\tilde{T} \cap \delta_u$ comes from $\mathfrak{f}_r$ for some r, as in Definition 4.1.10. There are $t_r - 1$ variables in $\delta_u \backslash \tilde{T}$, and thus there are at least t_r variables in $\delta_u \backslash S$. The product of these t_r variables is a minimal generator of $in(I_t(\tilde{F}))$ involving no elements of S, which contradicts our choice of S. Therefore no such S exists, and $|\tilde{T}|$ is indeed minimal. It thus follows that

$$ht\, I_t(\tilde{F}) = ht\, in(I_t(\tilde{F})) = |\tilde{T}| = q_0 + q_1 + \cdots + q_r$$

as claimed.

□

Corollary 4.1.13. *Let $I_t(\tilde{F})$, $I_{t'}(\tilde{G})$, and $I_\tau(\tilde{H})$ be defined as in Definition 4.1.6. Then*

$$ht\, I_t(\tilde{F}) = ht\, I_{t'}(\tilde{G}) = ht\, I_\tau(\tilde{H}) + 1.$$

Proof. Recall the convention that $t_0 = t_1$ and $a_0 = d_0 = a_1$. Let r be the index of the corner that is adjusted. In the height formula of Proposition 4.1.12 , we will use the notation $ht\, I_{t'}(\tilde{G}) = q_0' + q_1' + \cdots + q_k'$ and $ht\, I_\tau(\tilde{H}) = q_0'' + q_1'' + \cdots + q_r'' + \cdots + q_k''$, where $q_r'' = q_{r_1}'' + q_{r_2}''$ corresponds to the corners in H that replace the r^{th} corner of F. We consider two cases, which correspond to the two cases in the definition of H.

Case 1: Suppose $r = 1$ and $d_1 = n$, where n is the number of rows in F, so that the adjusted corner is the last entry on the diagonal and is the only entry of that row. Then in the formula for the height of $I_t(\tilde{F})$ we have that

$$q_0 = \binom{a_0 - t_0 + 2}{2}$$

and

$$q_1 = \max\{0, (a_1 - t_1 + 1)(d_1 - t_1 - d_0 + t_0)\}.$$

For $I_{t'}(\tilde{G})$ we have that

$$q_0' = \binom{(a_0 - 1) - (t_0 - 1) + 2}{2} = \binom{a_0 - t_0 + 2}{2} = q_0$$

and

$$\begin{aligned}
q_1' &= \max\{0, ((a_1-1)-(t_1-1)+1)((d_1-1)-(t_1-1)-(d_0-1)+(t_0-1))\} \\
&= \max\{0, (a_1-t_1+1)(d_1-t_1-d_0+t_0)\} \\
&= q_1.
\end{aligned}$$

Since $q_i' = q_i$ for $i > 1$, it follows that $ht\, I_t(\tilde{F}) = ht\, I_{t'}(\tilde{G})$.

For $ht\, I_\tau(\tilde{H})$, there are still only k corners, so $ht\, I_\tau(\tilde{H}) = q_0'' + q_1'' + \cdots + q_k''$. Recall that $\tau = t$. We have that

$$q_0'' = \binom{(a_0-1)-t_0+2}{2}$$

and

$$\begin{aligned}
q_1'' &= \max\{0, (a_1''-t_1''+1)(d_1''-t_1''-d_0''+t_0'')\} \\
&= ((a_1-1)-t_1+1)(d_0-t_0-(d_0-1)+t_0) \\
&= (a_1-t_1)(d_1-t_1-(a_1-1)+t_0) \\
&= (a_1-t_1)
\end{aligned}$$

The height formula for $I_t(\tilde{F})$ begins:

$$\begin{aligned}
q_0 + q_1 &= \binom{a_0-t_0+2}{2} + \max\{0, (a_1-t_1+1)(d_1-t_1-d_0+t_0)\} \\
&= \binom{a_0-t_0+2}{2} \\
&= \binom{a_0-t_0+1}{2} + (a_0-t_0+1)
\end{aligned}$$

The height formula for $I_\tau(\tilde{H})$ begins:

$$\begin{aligned}
q_0'' + q_1'' =& \binom{(a_0-1)-t_0+2}{2} + (a_1-t_1) \\
=& \binom{a_0-t_0+1}{2} + (a_0-t_0+1) - 1 \\
=& q_0 + q_1 - 1.
\end{aligned}$$

Since $q_i'' = q_i$ for $i > 1$, it follows that $ht\, I_t(\tilde{F}) = ht\, I_\tau(\tilde{H}) + 1$.

Case 2: Suppose that either $r > 1$ or that $r = 1$ and $d_1 > n$. Then the formula for the r^{th} summand in the height formula for $I_t(\tilde{F})$ is

$$q_r = (a_r - t_r + 1)(d_r - t_r - d_{r-1} + t_{r-1}).$$

The formula for the r^{th} summand in the height formula of $I_{t'}(\tilde{G})$ is

$$\begin{aligned} q'_r &= (a_r - 1) - (t_r - 1) + 1)((d_r - 1) - (t_r - 1) - d_{r-1} + t_{r-1}) \\ &= ((a_r - t_r + 1)(d_r - t_r - d_{r-1} + t_{r-1}) \\ &= q_r \end{aligned}$$

Since $q'_i = q_i$ for $i \neq r$, it follows that $ht\, I_t(\tilde{F}) = ht\, I_{t'}(\tilde{G})$.

In the height formula for $I_\tau(\tilde{H})$, we have that

$$q''_{r_1} = (a_r - t_r + 1)((d_r - 1) - t_r - d_{r-1} + t_{r-1})$$

and

$$q''_{r_2} = ((a_r - 1) - t_r + 1)(d_r - t_r - (d_r - 1) + t_r) = (a_r - t_r).$$

The sum of these is

$$\begin{aligned} q''_r = q''_{r_1} + q''_{r_2} &= (a_r - t_r + 1)(d_r - t_r - d_{r-1} + t_{r-1}) - 1 \\ &= q_r - 1. \end{aligned}$$

Since $q''_i = q_i$ for $i \neq r$, it follows that $ht\, I_t(\tilde{F}) = ht\, I_\tau(\tilde{H}) + 1$, as desired.

□

Liaison classification of reflected skew tableau ideals with no holes

Our goal now is to show that $I_t(\tilde{F})$ is linked to the target ideal $\tilde{\mathfrak{f}}$ generated by the indeterminates in the target region $\tilde{T}$. First, we recall that the initial ideal $in(I_t(\tilde{F}))$ is glicci, and thus that skew tableau ideals with no holes are Cohen-Macaulay. Then we show that the full ideals are glicci by producing an elementary biliaison between $I_t(\tilde{F})$ and $I_{t'}(\tilde{G})$ on $I_\tau(\tilde{H})$. This technique depends on the fact that the ideals are Cohen-Macaulay, and we re-prove this fact using an inductive approach that does not depend on knowing the initial ideals.

Theorem 4.1.14 ([12, Theorem 4.7])**.** *Let $\tilde{F}$ be a skew tableau with no holes. Then the initial ideal $in(I_t(\tilde{F}))$ is glicci.*

Corollary 4.1.15 ([12, Theorem 4.7])**.** *The ideal $in(I_t(\tilde{F}))$ is Cohen-Macaulay, and the associated simplicial complex is vertex-decomposable.*

We now provide a new proof that skew tableau ideals from tableaux with no holes are glicci.

Theorem 4.1.16 (c.f. [9, Theorem 2.4])**.** *Let F be a skew tableau with no holes. Then the ideal $I_t(\tilde{F})$ is glicci.*

Proof. Let $I_t(\tilde{F})$, $I_{t'}(\tilde{G})$, and $I_\tau(\tilde{H})$ be defined as in Definition 4.1.6, and let r be the index of the corner that is adjusted. We will show that

$$I_t(\tilde{F})/I_\tau(\tilde{H}) \cong I_{t'}(\tilde{G})/I_\tau(\tilde{H})\ (-1)$$

is an elementary biliaison between $I_t(\tilde{F})$ and $I_{t'}(\tilde{G})$. This graded ring isomorphism is given by elements f and g such that

$$f \cdot I_t(\tilde{F}) + I_\tau(\tilde{H}) = g \cdot I_{t'}(\tilde{G}) + I_\tau(\tilde{H}). \qquad (\star)$$

Let g be the minor from rows $\{1, 2, \ldots, t_r - 1, a_r\}$ and columns $\{d_r - t_r + 1, d_r - t_r + 2, \ldots, d_r\}$. Let f be the minor from rows $\{1, 2, \ldots, t_r - 1\}$ and columns $\{d_r - t_r + 1, d_r - t_r + 2, \ldots, d_r - 1\}$. Note that g is a $t_r \times t_r$ minor and f is the $(t_r - 1) \times (t_r - 1)$ minor using all but the last row and column of g, so that $\deg(f) - \deg(g) = -1$. As long as $I_{t_r}(\tilde{F}_r)$ is nonempty, the submatrices which form the minors f and g are subsets of $\tilde{F}$.

We will show $(\star)$ at the level of generators. For each generator $a \in I_t(\tilde{F})$, we choose a generator $b \in I_{t'}(\tilde{G})$ such that

$$f \cdot a - g \cdot b \in I_\tau(\tilde{H}).$$

The same argument will show that for any generator $b \in I_{t'}(\tilde{G})$, we can choose a generator $a \in I_t(\tilde{F})$ with the same result.

First we note that if $i \neq r$, then $I_{t_i}(\tilde{F}) = I_{t'_i}(\tilde{G}) \subset I_\tau(\tilde{H})$. Thus it is enough to show that

$$f \cdot I_{t_r}(\tilde{F}_r) + I_\tau(\tilde{H}) = g \cdot I_{t'_r}(\tilde{G}_r) + I_\tau(\tilde{H}).$$

Now, if a is any minor in $I_{t_r}(\tilde{F})$ not involving the variable x_{a_r,d_r}, then $a \in I_\tau(\tilde{H})$. Using the notation defined in Notation 2.3.1, let $M = \tilde{F}_r$ and let $a = M(I + a_r|J + d_r)$ be any minor in $I_{t_r}(\tilde{F}_r)$ involving x_{a_r,d_r}. This corresponds to $b = M(I|J) \in I_{t'_r}(\tilde{G}_r)$. With $g = M(K + a_r|L + d_r)$ and $f = M(K|L)$ defined as above, it follows from Corollary 2.3.11 that

$$f \cdot a - g \cdot b \in I_\tau(\tilde{H}).$$

In the reverse direction, any minor $b = M(I|J) \in I_{t'_r}(\tilde{G}_r)$ corresponds to a minor $a = M(I + a_r|J + d_r) \in I_{t_r}(\tilde{F}_r)$, and the same argument applies. This establishes the equality $(\star)$.

Skew tableau ideals with no holes are prime by Proposition 4.1.5, and so in particular, f and g are non zero-divisors of H. By Proposition 4.1.12, $ht\, I_t(\tilde{F}) = ht\, I_{t'}(\tilde{G}) = ht\, I_\tau(\tilde{H}) + 1$. By Corollary 4.1.15, the ideals are Cohen-Macaulay; however, we offer the following inductive argument, which may be useful in other settings where the initial ideals are not known. We use induction on N, the number of entries in the skew tableau. The case $N = 1$ is trivial. Assume that every skew tableau ideal generated from a tableau with N variables is Cohen Macaulay, and suppose $\tilde{F}$ has $N + 1$ entries. Note that both $\tilde{G}$ and $\tilde{H}$ have fewer entries than $\tilde{F}$, and so by assumption they are Cohen Macaulay. Thus $I_{t'}(\tilde{G})$ is obtained from $I_t(\tilde{F})$ by an elementary biliaison, and since $I_{t'}(\tilde{G})$ is Cohen-Macaulay by assumption, Theorem 2.1.6 gives that $I_t(\tilde{F})$ is Cohen-Macaulay also.

From the arguments above, we conclude that ($\star$) is an elementary biliaison, and thus it follows from Theorem 2.1.5 that the ideals $I_t(\tilde{F})$ and $I_{t'}(\tilde{G})$ are G-linked in two steps. Since the vector t' is formed from t by decreasing the r^{th} component t_r by 1, we may iterate this process $\sum_{r=1}^{k}(t_r - 1)$ times until we reach the target ideal $\mathfrak{f}$, which is generated by variables and is thus a complete intersection.

$\square$

4.2 General skew tableaux

When a skew tableau F has holes, it may no longer be the case that the minors form a Gröbner basis, and we do not have a codimension formula for these ideals. It is also not always clear how to define the tableaux G and H for the linkage. We will give examples of ideals for which the Gröbner basis property fails, and show some partial results for the linkage of skew tableau ideals in nice cases.

Symmetrized skew tableaux ideals are doubly symmetrized in the sense that the shape F is reflected to give $\tilde{F}$, and the indeterminates are specialized so that $\tilde{F}$ is a subset of a symmetric matrix. If we take F to be a subset of a generic matrix of indeterminates without specializing below the diagonal, then the ideal $I_t(\tilde{F})$ is not generally even Cohen-Macaulay.

Example 4.2.1. Let $\tilde{F}$ be the "unspecialized" tableau

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$
$x_{2,1}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$x_{3,1}$	$x_{3,2}$		$x_{3,4}$	$x_{3,5}$
$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$	$x_{4,5}$
$x_{5,1}$	$x_{5,2}$	$x_{5,3}$	$x_{5,4}$	

The shape is reflected but the variables are all distinct. Macaulay2 computes that $I_3(\tilde{F})$ is not Cohen-Macaulay.

Unsymmetrized skew tableau ideals

In this section, we discuss unsymmetrized skew tableau ideals from minors of mixed size. First, we observe that these ideals are equivalent in some cases to ladder determinantal ideals (which are known to be glicci), but in general the two classes of ideals are distinct. We then apply Theorem 2.3.9 to show that the set of minors generating the ideal forms a Gröbner basis.

Remark 4.2.2. Unsymmetrized skew tableau ideals generated by minors of a fixed size are equivalent to one-sided ladder determinantal ideals. However, if the ideals are generated by minors of mixed size, then these are a distinct class of ideals. To see this, let $F \subset M = (x_{i,j})$ be a skew tableau with r columns. For convenience we label these columns $1, \dots, r$. For this example, we drop the assumption that entries of F are on or above the main diagonal, and assume M is a generic matrix with exactly r columns. Define n_j to be the number of boxes in column j, and let σ be the permutation on $[r]$ such that

- $\sigma(p) < \sigma(q)$ if $n_p < n_q$, and
- $\sigma(p) < \sigma(q)$ if $n_p = n_q$ and $p < q$.

Then the column indices $\sigma(1), \dots, \sigma(r)$ are arranged in order of increasing number of boxes, and the order of columns with the same number of boxes is preserved. Let P_σ be the permutation matrix that arranges the columns of M in the order $\sigma(1), \dots, \sigma(r)$ via multiplication on the right. Let $\sigma(F)$ be the subset of MP_σ with contains the same entries as F. Then $\sigma(F)$ is a one-sided ladder. For instance, the columns of a tableau F would be rearranged as follows:

$$\begin{array}{cccccc}
x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} \\
x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} \\
x_{3,3} & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} \\
 & & x_{4,5} & x_{4,6} & x_{4,7} & x_{4,8} \\
 & & x_{5,5} & x_{5,6} & &
\end{array}
\quad \xrightarrow{\sigma} \quad
\begin{array}{cccccc}
x_{1,3} & x_{1,4} & x_{1,7} & x_{1,8} & x_{1,5} & x_{1,6} \\
x_{2,3} & x_{2,4} & x_{2,7} & x_{2,8} & x_{2,5} & x_{2,6} \\
x_{3,3} & x_{3,4} & x_{3,7} & x_{3,8} & x_{3,5} & x_{3,6} \\
 & & x_{4,7} & x_{4,8} & x_{4,5} & x_{4,6} \\
 & & & & x_{5,5} & x_{5,6}
\end{array}$$

The ideal generated by minors of fixed size in F is the same as the ideal generated by minors of the same size in $\sigma(F)$. However, our class of ideals differs from one-sided ladder ideals in two ways. First, we specify a diagonal term-order when studying the initial ideals, whereas one-sided ladder ideals have been studied previously using anti-diagonal term orders. Second we allow minors of mixed size, where the size of the minors is determined by right corners (shaded in gray above). It may happen that σ takes a right corner of F to a non-corner of $\sigma(F)$. When this happens, the ideal does not satisfy the conditions to be a mixed ladder determinantal ideal. See, for example, [7] and [8] for results about the linkage of mixed ladder determinantal ideals.

Theorem 4.2.3. *Fix the lexicographic term order. Let $F = \bigcup_{i=1}^k F_i$ be a skew tableau with associated vector $t = (t_1, \dots, t_k)$ and let $\mathfrak{G}_i$ be the set of $t_i \times t_i$ minors in F_i. Then $\mathfrak{G} = \bigcup \mathfrak{G}_i$ is a Gröbner basis of $I_t(F)$.*

Proof. We will show that the S-polynomial of a pair of minors in $\mathfrak{G}$ reduces to zero with respect to $\mathfrak{G}$. By Remark 4.2.2 and [8], $\mathfrak{G}_i$ is a Gröbner basis of $I_{t_i}(F_i)$, so if $f, g \in \mathfrak{G}_i$, $S(f,g)$ reduces to zero with respect to $\mathfrak{G}_i$. Suppose that $f \in \mathfrak{G}_i$ and $g \in \mathfrak{G}_j$. To avoid unnecessary indices, we will write r for t_i and s for t_j. We assume without loss of generality that $r \leq s$. We also assume that $i > j$, since otherwise

$g \in I_r(F_i)$ and so g is redundant.

Let $p \in \{0, 1, \ldots, r\}$ be the number of entries that f and g have in common on the main diagonals of the submatrices they come from. We consider these as submatrices of a matrix M of size $n \times n$ with $n = (r + s - p)$, where f comes from the upper left $r \times r$ submatrix of M and g comes from the lower right $s \times s$ submatrix of M. For example, if $r = 3$, $s = 4$, and $p = 2$, the matrix M is:

$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$
$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$
$x_{3,3}$	$x_{3,4}$	$x_{3,5}$	$x_{3,6}$	$x_{3,7}$
	$x_{4,4}$	$x_{4,5}$	$x_{4,6}$	$x_{4,7}$
	$x_{5,4}$	$x_{5,5}$	$x_{5,6}$	$x_{5,7}$

We have omitted the entries which will not be used in our calculations, and shaded in gray the entries that f and g share on their main diagonals. Here $f = M(1, 2, 3|1, 2, 3)$ and $g = M(2, 3, 4, 5|2, 3, 4, 5)$, and in general $f = M(1, 2, \ldots, r|1, 2, \ldots, r)$ and $g = M(r - p + 1, r - p + 2 \ldots, n|r - p + 1, r - p + 2 \ldots, n)$.

If $p = 0$, then the leading terms of f and g are relatively prime, and so it is well-known (see, for example, [14, Lemma 2.3.1]) that their S-polynomial reduces to zero with respect to $\{f, g\}$. Suppose that $1 \leq p \leq r$. Then

$$\begin{aligned} S(f,g) &= \frac{\operatorname{lcm}(in(f), in(g))}{in(f)} \cdot f - \frac{\operatorname{lcm}(in(f), in(g))}{in(g)} \cdot g \\ &= x_{r+1,r+1} x_{r+2,r+2} \ldots x_{n,n} \cdot f - x_{1,1} x_{2,2} \ldots x_{r-1,r-1} \cdot g. \end{aligned}$$

We wish to show that $S(f, g)$ reduces to zero with respect to $\mathfrak{G}$.

Fix $A = \{1, 2, \ldots, r - p\}$ and $B = \{1, 2, \ldots, r\}$. We apply Theorem 2.3.9 to the transpose of M to get

$$\sum_{V \subseteq B} M\{V|A\} = \sum_{U \supseteq A} M\{B|U\} \qquad (\star)$$

where U and V are subsets of $[n]$. If we label the minors $M(\tilde{V}|\tilde{A})$ as f_p for $p = 1, \ldots, \binom{|B|}{|V|}$, then there are coefficients α_p such that

$$\sum_{V \subseteq B} M\{V|A\} = x_{r+1,r+1} x_{r+2,r+2} \ldots x_{n,n} \cdot f + \sum \alpha_p \cdot f_p.$$

Note $f_p = f$ for some p. Similarly, if we label the minors $M(B|U)$ as g_q, where $q = 1, \ldots, \binom{|U|}{|A|}$, then there are coefficients β_q such that

$$\sum_{U \supseteq A} M\{B|U\} = x_{1,1} x_{2,2} \ldots x_{r-1,r-1} \cdot g + \sum \beta_q \cdot g_q.$$

Note $g_q = g$ for some q. Then the equation ($\star$) becomes

$$\begin{aligned} & x_{r+1,r+1}x_{r+2,r+2}\dots x_{n,n}\cdot f + \sum \alpha_p \cdot f_p = x_{1,1}x_{2,2}\dots x_{r-1,r-1}\cdot g + \sum \beta_q \cdot g_q \\ \Longrightarrow\ & x_{r+1,r+1}x_{r+2,r+2}\dots x_{n,n}\cdot f - x_{1,1}x_{2,2}\dots x_{r-1,r-1}\cdot g = \sum \beta_q \cdot g_q - \sum \alpha_p \cdot f_p \\ \Longrightarrow\ & S(f,g) = \sum \beta_q \cdot g_q - \sum \alpha_p \cdot f_p \end{aligned}$$

We claim that this gives a reduction of $S(f,g)$ to zero with respect to $\mathfrak{G}$.

Note that the leading term of a minor $M(a_1,\dots,a_t|b_1,\dots,b_t)$ is the monomial $x_{a_1,b_1}\cdot x_{a_2,b_2}\cdot\dots\cdot x_{a_t,b_t}$. To find the leading term of $S(f,g)$, we observe that

$$\begin{aligned} \frac{\mathrm{lcm}(in(f),in(g))}{in(f)}\cdot f = {} & x_{1,1}x_{2,2}\dots x_{n,n} \\ & - x_{1,1}x_{2,2}\dots x_{r-p,r-p}x_{r-p,r-p+1}x_{r-p+1,r-p}x_{r-p+2,r-p+2},\dots,x_{n,n} \\ & + \dots \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{lcm}(in(f),in(g))}{in(g)}\cdot g = {} & x_{1,1}x_{2,2}\dots x_{n,n} \\ & - x_{1,1}x_{2,2}\dots x_{n-2,n-2}x_{n-1,n}x_{n,n-1} \\ & + \dots \end{aligned}$$

The terms are written in decreasing order, and so it follows that

$$in(S(f,g)) = x_{1,1}x_{2,2}\dots x_{n-2,n-2}x_{n-1,n}x_{n,n-1}.$$

The largest monomials that appear in the equation ($\star$) are $x_{1,1}x_{2,2}\dots x_{n,n}$ and $x_{1,1}x_{2,2}\dots x_{n-2,n-2}x_{n-1,n}x_{n,n-1}$. The former appears on the left hand side when $V = \{1,2,\dots,r-p\}$ as the leading term of

$$M(1,\dots,r-p|1,\dots,r-p)\cdot M(r-p+1,\dots,n|r-p+1,\dots,n)$$

and on the right hand side when $U = \{1,2,\dots,r\}$ as the leading term of

$$M(1,2,\dots,r|1,2,\dots,r)\cdot M(r+1,r+2,\dots,n|r+1,r+2,\dots,n).$$

The latter term appears only on the left hand side, as the second term of the summand with $V = \{1,2,\dots,r-p\}$. From this it follows that the largest leading term of any $\alpha_i \cdot f_i$ or $\beta_j \cdot g_j$ is

$$x_{1,1}x_{2,2}\dots x_{n-2,n-2}x_{n-1,n}x_{n,n-1},$$

which is exactly the leading term of $S(f,g)$.

By our choice of A and B and the definition of the matrix M, all terms in equation $(\star)$ are present in $K[F]$ by the definition of a skew tableau. Since $|\tilde{A}| = s$, it follows that $f_p \in \mathfrak{G}_i$ for each p, and since $|B| = r$, it follows that $g_q \in \mathfrak{G}_j$ for each q. Thus

$$S(f,g) = \sum \alpha_p \cdot f_p - \sum \beta_q \cdot g_q$$

is a reduction of $S(f,g)$ to zero with respect to $\mathfrak{G}$. By Theorem 2.2.4, $\mathfrak{G}$ is therefore a Gröbner basis of $I_t(F)$.

$\square$

Example 4.2.4. For unsymmetrized skew tableau ideals generated by minors of size 3×3 or smaller, we consider the S-polynomial of f and g in three cases. Let $\mathfrak{G}$ be the set of minors generating $I_t(F)$.

Case 1: Suppose f and g are the determinants of 3×3 matrices which overlap in a 2×2 submatrix on the main diagonal of both matrices. We may assume, without loss of generality, that these come from a subregion of F of the following form:

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$
$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$
	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$

where $f = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix}$ and $g = \begin{vmatrix} x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,2} & x_{4,3} & x_{4,4} \end{vmatrix}$. Their S-polynomial is

$$S(f,g) = x_{4,4} \cdot f - x_{1,1} \cdot g.$$

Division of $S(f,g)$ by $\mathfrak{G}$ yields the expression

$$\begin{aligned} S(f,g) = &- x_{2,1} \cdot \begin{vmatrix} x_{1,2} & x_{1,3} & x_{1,4} \\ x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,2} & x_{4,3} & x_{4,4} \end{vmatrix} + x_{3,2} \cdot \begin{vmatrix} x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,2} & x_{2,3} & x_{2,4} \\ x_{4,2} & x_{4,3} & x_{4,4} \end{vmatrix} \\ &- x_{4,2} \cdot \begin{vmatrix} x_{1,1} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,3} & x_{3,4} \end{vmatrix} + x_{4,3} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,4} \end{vmatrix}. \end{aligned}$$

gives the division of $S(f,g)$ by $\mathfrak{G}$ with remainder zero.

Case 2: Suppose f and g are the determinants of 3×3 matrices which overlap in a single entry on the main diagonal of both matrices. We may assume, without loss of generality, that these come from a subregion of F of the following form:

$$
\begin{array}{|c|c|c|c|c|}
\hline
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\
\hline
x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\
\hline
x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\
\hline
\multicolumn{2}{c|}{} & x_{4,3} & x_{4,4} & x_{4,5} \\
\cline{3-5}
\multicolumn{2}{c|}{} & x_{5,3} & x_{5,4} & x_{5,5} \\
\cline{3-5}
\end{array}
$$

where $f = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix}$ and $g = \begin{vmatrix} x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,3} & x_{5,4} & x_{5,5} \end{vmatrix}$. Their S-polynomial is

$$S(f,g) = x_{4,4}x_{5,5} \cdot f - x_{1,1}x_{2,2} \cdot g.$$

Division of $S(f,g)$ by $\mathfrak{G}$ yields the expression

$$
\begin{aligned}
S(f,g) = &- x_{1,2}x_{2,1} \cdot \begin{vmatrix} x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,3} & x_{5,4} & x_{5,5} \end{vmatrix} - \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{3,1} & x_{3,2} \end{vmatrix} \cdot \begin{vmatrix} x_{2,3} & x_{2,4} & x_{2,5} \\ x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,3} & x_{5,4} & x_{5,5} \end{vmatrix} \\
&+ \begin{vmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{vmatrix} \cdot \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,3} & x_{5,4} & x_{5,5} \end{vmatrix} - \begin{vmatrix} x_{4,3} & x_{4,4} \\ x_{5,3} & x_{5,4} \end{vmatrix} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,5} \end{vmatrix} \\
&+ \begin{vmatrix} x_{4,3} & x_{4,5} \\ x_{5,3} & x_{5,5} \end{vmatrix} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,4} \end{vmatrix} - x_{4,5}x_{5,4} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix}.
\end{aligned}
$$

This gives the division of $S(f,g)$ by $\mathfrak{G}$ with remainder zero.

Case 3: Suppose f is the determinants of a 2×2 submatrix and g is the determinant of a 3×3 matrix, and that they overlap in a single entry on the main diagonal of both matrices. We may assume, without loss of generality, that these come from a subregion of F of the following form:

$$
\begin{array}{|c|c|c|c|}
\hline
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\
\hline
x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\
\hline
\multicolumn{1}{c|}{} & x_{3,2} & x_{3,3} & x_{3,4} \\
\cline{2-4}
\multicolumn{1}{c|}{} & x_{4,2} & x_{4,3} & x_{4,4} \\
\cline{2-4}
\end{array}
$$

where $f = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix}$ and $g = \begin{vmatrix} x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,2} & x_{4,3} & x_{4,4} \end{vmatrix}$. Their S-polynomial is

$$x_{3,3} \cdot f - x_{1,1} \cdot g.$$

Division of $S(f,g)$ by $\mathfrak{G}$ yields the expression

$$S(f,g) = -\begin{vmatrix} x_{3,2} & x_{3,3} \\ x_{4,2} & x_{4,3} \end{vmatrix} \cdot \begin{vmatrix} x_{1,1} & x_{1,4} \\ x_{2,1} & x_{2,4} \end{vmatrix} + \begin{vmatrix} x_{3,2} & x_{3,4} \\ x_{4,2} & x_{4,4} \end{vmatrix} \cdot \begin{vmatrix} x_{1,1} & x_{1,3} \\ x_{2,1} & x_{2,3} \end{vmatrix}$$
$$+ x_{3,4}x_{4,3} \cdot \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix} - x_{2,1} \cdot \begin{vmatrix} x_{1,2} & x_{1,3} & x_{1,4} \\ x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,2} & x_{4,3} & x_{4,4} \end{vmatrix}.$$

Again, this gives the division of $S(f,g)$ by $\mathfrak{G}$ with remainder zero.

Remark 4.2.5. We hope to use this Gröbner basis to show that the initial ideals of unsymmetrized skew tableau ideals are glicci. The Gröbner basis should also facilitate codimension computations, primality testing, and showing that the full ideals are glicci.

Skew tableaux with "nice holes"

Definition 4.2.6. Let $F = T(a,b,c,d)$ be a skew tableau such that

$$\{(i,j) \,|\, 1 \le i \le a_r \text{ and } d_r - t_r + 1 \le j \le d_r\} \subset \tilde{F}.$$

We call $\tilde{F}$ a *symmetrized skew tableau with nice holes.*

Proposition 4.2.7. *Let F be a nice skew tableau ideal with G and H defined as in Definition 4.1.6. Then there exist homogeneous polynomials f and g such that*

$$f \cdot I_t(\tilde{F}) + I_\tau(\tilde{H}) = g \cdot I_{t'}(\tilde{G}) + I_\tau(\tilde{H}).$$

Proof. Let g be the minor from rows $\{1,2,\ldots,t_r-1,a_r\}$ and columns $\{d_r-t_r+1, d_r-t_r+2,\ldots,d_r\}$. Let f be the minor from rows $\{1,2,\ldots,t_r-1\}$ and columns $\{d_r-t_r+1, d_r-t_r+2,\ldots,d_r-1\}$. Using the notation defined in Notation 2.3.1, let $M = X_r$ and let $a = M(I+a_r|J+d_r)$ be any minor in $I_{t_r}(\tilde{F}_r)$ involving x_{a_r,d_r}. This corresponds to $b = M(I|J) \in I_{t'_r}(\tilde{G}_r)$. We use $g = M(K+a_r|L+d_r)$ and $f = M(K|L)$ defined as above, and note that these minors are in $I_t(\tilde{F})$ by the definition of a skew tableau with nice holes. We wish to show that

$$f \cdot a - g \cdot b \in I_\tau(\tilde{H})$$

for the corresponding elements a and b as in the proof of Theorem 4.1.16. Although this expression fits the pattern of Corollary 2.3.11, we must verify that all of the submatrices of M that are used in the proof are present in $\tilde{F}$. Once we have done this, we note as before that any minor $b = M(I|J) \in I_{t'_r}(\tilde{G}_r)$ corresponds to a minor $a = M(I+a_r|J+d_r) \in I_{t_r}(\tilde{F}_r)$, and the same argument will applies in the reverse direction.

Recall from Proposition 2.3.10 that

$$\begin{aligned}M(I+m|J+n)\cdot M(K|L)-M(I|J)\cdot M(K+m|L+n)&\\ =\sum_{i\in I}(-1)^{n+i}\,M(I-i+m|J)\cdot M(K+i|L+n)&\\ -\sum_{l\in L}(-1)^{n-l}M(I+m|J+l)\cdot M(K|(L-l+n),&\end{aligned}$$

The left-hand side of the equation corresponds to the expression $f\cdot a-g\cdot b$. In the summations on the right-hand side, we need to verify that $M_{(K+i|L+n)}$ and $M_{(I+m|J+l)}$ are subsets of $\tilde{F}$. To do this, we only need to check that $(i,l)\in\tilde{F}$ for any $i\in I$ and $l\in L$. In this case, by the choice of I we know that $1\le i\le a_r$, and by the choice of L we know that $d_r-t_r+1\le j\le d_r$. By definition of a tableau with nice holes, it follows that $(i,l)\in\tilde{F}$, and since $(i,l)\ne(a_r,d_r)$, in fact $(i,l)\in\tilde{H}$. Thus the conclusion of Corollary 2.3.11 applies, and so

$$f\cdot I_t(\tilde{F})+I_\tau(\tilde{H})=g\cdot I_{t'}(\tilde{G})+I_\tau(\tilde{H}).$$

□

Remark 4.2.8. (1.) In order to have basic double links in the preceding proposition, it remains to show that the chosen elements f and g are nonzero-divisors of $I_\tau(\tilde{H})$.

(2.) In order to give an elementary biliaison in the preceding proposition, we would further need to show that $I_\tau(\tilde{H})$ is Cohen-Macaulay and generically Gorenstein.

Our argument in Section 4.1 uses an inductive argument based on the fact that if $\tilde{F}$ does not have holes, then neither does $\tilde{G}$ or $\tilde{H}$. If $\tilde{F}$ is a skew tableau with nice holes, then it still follows that $\tilde{G}$ has nice holes. However, it does not necessarily follow that $\tilde{H}$ has nice holes, and in fact if there is even a single hole in $\tilde{F}$, then any inductive argument we use depends on linking ideals from tableaux that do not have nice holes.

Skew tableaux with arbitrary holes

Symmetrized skew tableau ideals in general do not have the nice properties we have observed in special cases. For example, the minors do not usually form a Gröbner basis, and the ideals may not be prime. On the other hand, we have not been able to construct an example which fails to be Cohen-Macaulay. This offers a ray of hope that the ideals may still have nice structure from the perspective of liaison theory.

Example 4.2.9. For arbitrary symmetrized skew tableau ideals, the set of minors may not be a Gröbner basis. Let $\tilde{F}$ be the tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\
\hline
x_{1,2} & & x_{2,3} & x_{2,4} & x_{2,5} \\
\hline
x_{1,3} & x_{2,3} & & x_{3,4} & x_{3,5} \\
\hline
x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} \\
\cline{1-4}
x_{1,5} & x_{2,5} & x_{3,5} \\
\cline{1-3}
\end{array}$$

The two minors generating $I_3(\tilde{F})$ are

$$\begin{vmatrix} x_{1,1} & x_{1,4} & x_{1,5} \\ x_{1,2} & x_{2,4} & x_{2,5} \\ x_{1,3} & x_{3,4} & x_{3,5} \end{vmatrix}, \begin{vmatrix} x_{1,1} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,3} & x_{2,4} \\ x_{1,4} & x_{3,4} & x_{4,4} \end{vmatrix}$$

However, the initial ideal $in(I_3(\tilde{F}))$ is generated by the three terms

$$\begin{aligned} &x_{1,1}x_{2,4}x_{3,5} \\ &x_{1,1}x_{2,3}x_{4,4} \\ &x_{1,2}x_{1,3}x_{2,4}x_{3,5}x_{4,4} \end{aligned}$$

The degree five term comes, for example, as the leading term of

$$\begin{vmatrix} x_{1,1} & x_{1,4} & x_{1,5} \\ x_{1,2} & x_{2,4} & x_{2,5} \\ x_{1,3} & x_{3,4} & x_{3,5} \end{vmatrix} \cdot \begin{vmatrix} x_{2,3} & x_{2,4} \\ x_{3,4} & x_{4,4} \end{vmatrix} - \begin{vmatrix} x_{1,1} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,3} & x_{2,4} \\ x_{1,4} & x_{3,4} & x_{4,4} \end{vmatrix} \cdot \begin{vmatrix} x_{2,4} & x_{2,5} \\ x_{3,4} & x_{3,5} \end{vmatrix}.$$

Example 4.2.10. Even for symmetrized skew tableau ideals from tableaux with nice holes, the set of minors may not be a Gröbner basis. Let $\tilde{F}$ be the tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\
\hline
x_{1,2} & & x_{2,3} & x_{2,4} & x_{2,5} \\
\hline
x_{1,3} & x_{2,3} & & x_{3,4} & x_{3,5} \\
\hline
x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} & x_{4,5} \\
\hline
x_{1,5} & x_{2,5} & x_{3,5} & x_{4,5} & x_{5,5} \\
\hline
\end{array}$$

Macaulay2 gives 16 minimal generators of $I_3(\tilde{F})$. It gives the 16 main diagonals of these minors as minimal generators for $in(I_3(\tilde{F}))$, along with the degree 5 term $x_{1,2}x_{1,3}x_{2,4}x_{3,5}x_{4,4}$. This is the same degree 5 term as in the previous example.

Example 4.2.11. Arbitrary symmetrized skew tableau ideals may not be prime. Consider the following skew tableau:

	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$
$x_{1,2}$		$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$
$x_{1,3}$	$x_{2,3}$		$x_{3,4}$	$x_{3,5}$	$x_{3,6}$
$x_{1,4}$	$x_{2,4}$	$x_{3,4}$	$x_{4,4}$	$x_{4,5}$	$x_{4,6}$
$x_{1,5}$	$x_{2,5}$	$x_{3,5}$	$x_{4,5}$		
$x_{1,6}$	$x_{2,6}$	$x_{3,6}$	$x_{4,6}$		

Macaulay2 calculates that the minor $f = M(1,3,4|3,4,5)$ is a zero divisor modulo H, where H is the ideal generated by 3×3 minors in the skew tableau which do not include the entry $x_{4,6}$.

4.3 Future work

There are still some open questions about symmetric skew tableau ideals that we would like to resolve. Our main project right now is to show that ideals generated by $t \times t$ minors in a skew tableau are glicci for $t \geq 3$. We also hope to obtain a nice description of a Gröbner basis and use this to extend the linkage to the initial ideals. The first line of inquiry in this direction will be to discern the relationship between the presence of "holes" caused by reflection and the failure of the minors to be a Gröbner basis. In the unsymmetrized case, our computation of a Gröbner basis provides a nice tool to compute codimension, test for primality, and show that the initial ideals are glicci. It is hoped that these results would also lead to a similar result for the full ideals.

When generating ideals from minors in some subset of a matrix of indeterminates, an arbitary shape may not give rise to particularly nice ideals; for symmetric skew-tableaux in which the entries are not symmetrized, the ideals are not even Cohen Macaulay, as we showed in Example 4.2.1. It is natural to ask, for which shapes are the ideals generated by minors glicci? Following [13], in which it was shown that 2×2 minors in a "simple polyomino" generate Cohen-Macaulay ideals, we would like to use liaison techniques to study these polyomino determinantal ideals and determine whether they are glicci.

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